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THE TRANSFORMS AND APPLICATIONS HANDBOOK

SECOND EDITION

Editor-in-Chief

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Preface

The purpose of *The Transforms and Applications Handbook, Second Edition* is to include in a single volume the most important mathematical transforms frequently used by engineers and scientists. The book also was written with the advanced undergraduate and graduate students in mind. Each chapter covers one of the transforms, accompanied by a number of examples that are included to elucidate the use of the transform and its properties. Applications to difference areas are included in each chapter as well. This inclusion gives readers of different backgrounds the opportunity to become familiar with the wide spectrum of applications of these transforms. We believe that having all of these useful transforms included in one book will be of great value to scientists, engineers, and students.

The information is now organized into 13 chapters, each covering one of the transforms, except for Chapter 1 which enhances some topics that are treated less extensively in the other chapters. Over the past 5 years, a number of communications have been received concerning different aspects of the Handbook. All of the comments regarding typographical errors have been incorporated in the second edition. The editor and the contributors wish to thank the readers for their contributions and encouragement which prompted this second edition.

In the second edition to the Handbook, we have added three new chapters: *Lapped Transforms*, *Discrete Time and Discrete Fourier Transforms*, and *Fractional Fourier Transforms*. In the original chapters, we have corrected typographical errors, replaced the table of Laplace transforms with another table containing a larger number of entries, the chapter on Mellin transforms was rewritten, the cosine and sine transforms were rewritten, and the Wavelet transforms were updated.

The Editor would be extremely grateful if the readers forwarded their opinion about the Handbook, any errors they may detect, suggestions for new material in new editions, and material that they feel may be neglected. The reader also may consult the following references:

1. Yu. A. Brychkov and A.P. Prudnikov, *Integral Transforms of Generalized Functions*, Gordon and Breach, 1989.
2. D.G. Duffy, *Transform Methods for Solving Partial Differential Equations*, CRC Press, Boca Raton, FL, 1994.
3. I.S. Gradshteyn and I.M. Ryzhik, *Tables of Integrals, Series and Products*, Academic Press, New York, 1980.
4. F. Oberhettinger, *Tables of Fourier Transforms and Fourier Transforms of Distributions*, Springer-Verlag, 1990.
5. A.P. Prudnikov, Yu.A. Brychkov, and O.I. Marichev, *Integrals and Series; Direct Laplace Transforms*, Vol. 4; *Inverse Laplace Transforms*, Vol. 5, Gordon and Breach, 1992.

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Alexander D. Poularikas, as a Fulbright scholar, attended the University of Arkansas where he received his Ph.D. in Electrical Engineering in 1966. He joined the University of Rhode Island as an assistant professor of Electrical and Computer Engineering the same year and became full professor in 1974. Poularikas joined the University of Denver as chairman of the Department of Engineering in 1985, and 2 years later moved to the University of Alabama in Huntsville as chairman of its Department of Electrical and Computer Engineering.

Poularikas has visited and done scientific work at the Massachusetts Institute of Technology, the Underwater Systems Center at Newport, Goddard Spaceflight Center in Maryland, and at Stanford University. He is a senior member of IEEE, a charter member of the Arkansas Academy of Electrical Engineers, a member of Tau Beta Pi, Sigma Xi, Sigma Pi Sigma, and received the Outstanding Educator Award from the IEEE Huntsville Section in both 1990 and 1996. He also is a member of numerous professional societies.

He has published more than 60 papers in scientific and engineering magazines and published the following books:

Electromagnetics, Marcel Dekker, 1979
Electrical Engineering: Introduction and Concepts, Matrix Publishers, 1982
Workbook, Matrix Publishers, 1982
Signals and Systems, Brooks/Cole, 1985
Elements of Signals and Systems, PWS-Kent, 1988
Signals and Systems, 2nd edition, PWS-Kent, 1992
The Transforms and Applications Handbook, CRC Press, 1995
The Handbook of Formulas and Tables for Signal Processing, CRC Press, 1998

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Contents

Poularikas, A.D. "Signals and Systems."
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1
Signals and Systems

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1.1 Introduction to Signals

A knowledge of a broad range of signals is of practical importance in describing human experience. In engineering systems, signals may carry information or energy. The signals with which we are concerned may be the cause of an event or the consequence of an action.

The characteristics of a signal may be of a broad range of shapes, amplitudes, time duration, and perhaps other physical properties. In many cases, the signal will be expressed in analytic form; in other cases, the signal may be given only in graphical form.

It is the purpose of this chapter to introduce the mathematical representation of signals, their properties, and some of their applications. These representations are in different formats depending on whether the signals are periodic or truncated, or whether they are deduced from graphical representations.

Signals may be classified as follows:

1. Phenomenological classification is based on the evolution type of signal, that is, a perfectly predictable evolution defines a deterministic signal and a signal with unpredictable behavior is called a **random signal**.
2. Energy classification separates signals into **energy signals**, those having finite energy, and **power signals**, those with a finite average power and infinite energy.
3. Morphological classification is based on whether signals are continuous, quantitized, sampled, or digital signals.
4. Dimensional classification is based on the number of independent variables.
5. Spectral classification is based on the shape of the frequency distribution of the signal spectrum.

1.1.1 Functions (Signals), Variables, and Point Sets

The **rule of correspondence** from a set S_x of real or complex number x to a real or complex number

$$y = f(x) \tag{1.1.1}$$

is called a function of the argument x. Equation (1.1.1) specifies a value (or values) y of the variable y (set of values in Y) corresponding to each suitable value of x in X. In (1.1.1) x is the **independent** variable and y is the **dependent** variable.

A function of n variables $x_1, x_2, \ldots, x_n$ associates values

$$y = f(x_1, x_2, \ldots, x_n) \tag{1.1.2}$$

of a dependent variable y with ordered sets of values of the independent variables $x_1, x_2, \ldots, x_n$.

The set S_x of the values of x (or sets of values of $x_1, x_2, \ldots, x_n$) for which the relationships (1.1.1) and (1.1.2) are defined constitutes the **domain** of the function. The corresponding set of S_y of values of y is the S_y **range** of the function.

A **single-valued** function produces a single value of the dependent variable for each value of the argument. A **multiple-valued** function attains two or more values for each value of the argument.

The function $y(x)$ has an **inverse** function $x(y)$ if $y = y(x)$ implies $x = x(y)$.

A function $y = f(x)$ is **algebraic** of x if and only if x and y satisfy a relation of the form $F(x, y) = 0$, where $F(x, y)$ is a polynomial in x and y. The function $y = f(x)$ is rational if $f(x)$ is a polynomial or is a quotient of two polynomials.

A real or complex function $y = f(x)$ is bounded on a set S_x if and only if the corresponding set S_y of values y is bounded. Furthermore, a real function $y = f(x)$ has an upper **bound**, **least upper bound**, **lower bound**, **greatest lower bound**, **maximum**, or **minimum** on S_x if this is also true for the corresponding set S_y.

Neighborhood

Given any finite real number a, an open neighborhood of the point a is the set of all points $\{x\}$ such that $|x - a| < \delta$ for any positive real number δ.

An open neighborhood of the point $(a_1, a_2, \ldots, a_n)$, where all a_i are finite, is the set of all points $(x_1, x_2, \ldots, x_n)$ such that $|x_1 - a_1| < \delta$, $|x_2 - a_2| < \delta$, … and $|x_n - a_n| < \delta$ for some positive real number δ.

Open and Closed Sets

A point P is a **limit** point (accumulation point) of the point set S if and only if every neighborhood of P has a neighborhood contained entirely in S, other than P itself.

A limit point P is an interior point of S if and only if P has a neighborhood contained entirely in S. Otherwise P is a **boundary** point.

A point P is an isolated point of S if an only if P has a neighborhood in which P is the only point belonging to S.

A point set is **open** if and only if it contains only interior points.

A point set is **closed** if and only if it contains all its limit points; a **finite** set is closed.

1.1.2 Limits and Continuous Functions

1. A single-value function $f(x)$ has a limit

$$\lim_{x \to a} f(x) = L, \quad L = \text{finite}$$

as $x \to a$ $[f(x) \to L$ as $x \to a]$ if and only if for each positive real number ϵ there exists a real number δ such that $0 < |x - a| < \delta$ implies that $f(x)$ is defined and $|f(x) - L| < \epsilon$.

2. A single-valued function $f(x)$ has a limit

$$\lim_{x\to\infty} f(x) = L, \quad L = \text{finite}$$

as $x \to \infty$ if and only if for each positive real number ε there exists a real number N such that $x > N$ implies that $f(x)$ is defined and $|f(x) - L| < \varepsilon$.

Operations with Limits

If limits exist, Table 1.2.1 gives the limit operations.

TABLE 1.2.1 Operations with Limits

$\lim_{x\to a}[f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
$\lim_{x\to a}[b\, f(x)] = b \lim_{x\to a} f(x)$
$\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \lim_{x\to a} g(x)$
$\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \qquad \left(\lim_{x\to a} g(x) \neq 0\right)$

a = may be finite or infinite

Asymptotic Relations Between Two Functions

Given two real or complex functions $f(x)$, $g(x)$ of a real or complex variable x, we write

1. $f(x) = O[g(x)]$; $f(x)$ is **of the order** $g(x)$ as $x \to a$ if and only if there is a neighborhood of $x = a$ such that $|f(x)/g(x)|$ is bounded.
2. $f(x) \sim g(x)$; $f(x)$ is **asymptotically proportional** to $g(x)$ as $x \to a$ if and only if $\lim_{x\to a} [f(x)/g(x)]$ exists and it is not zero.
3. $f(x) \cong g(x)$; $f(x)$ is **asymptotically equal** to $g(x)$ as $x \to a$ if and only if

$$\lim_{x\to a}\left[f(x)/g(x)\right] = 1.$$

4. $f(x) = o[g(x)]$; $f(x)$ becomes negligible compared with $g(x)$ if and only if

$$\lim_{x\to a}\left[f(x)/g(x)\right] = 0.$$

5. $f(x) = \varphi(x) + O[g(x)]$ if $f(x) - \varphi(x) = O[g(x)]$
   $f(x) = \varphi(x) + o[g(x)]$ if $f(x) - \varphi(x) = o[g(x)]$

Uniform Convergence

1. A single-valued function $f(x_1, x_2)$ **converges uniformly** on a set S of values of x_2, $\lim_{x_1\to a} f(x_1, x_2) = L(x_2)$ if and only if for each positive real number ε there exists a real number δ such that $0 < |x_1 - a| < \delta$ implies that $f(x_1, x_2)$ is defined and $|f(x_1, x_2) - L(x_2)| < \varepsilon$ for all x_2 in S (δ is independent of x_2).
2. A single-valued function $f(x_1, x_2)$ **converges uniformly** on a set S of values of x_2, $\lim_{x_1\to\infty} f(x_1, x_2) = L(x_2)$ if and only if for each positive real number ε there exists a real number N such that for $x_1 > N$ implies that $f(x_1, x_2)$ is defined and $|f(x_1, x_2) - L(x_2)| < \varepsilon$ for all x_2 in S.

5. A **sequence** of functions $f_1(x)$, $f_2(x)$, … **converges uniformly** on a set S of values of x to a finite and unique function

$$\lim_{n\to\infty} f_n(x) = f(x)$$

if and only if for each positive real number ε there exists a real integer N such that for $n > N$ implies that $|f_n(x) - f(x)| < \varepsilon$ for all n in S.

Continuous Functions

1. A single-valued function $f(x)$ defined in the neighborhood of $x = a$ is **continuous** at $x = a$ if and only if for every positive real number ε there exists a real number δ such that $|x - a| < \delta$ implies $|f(x) - f(a)| < \varepsilon$.
2. A function is **continuous on a series of points** (interval or region) if and only if it is continuous at each point of the set.
3. A real function continuous on a bounded closed interval $[a, b]$ is bounded on $[a, b]$ and assumes every value between and including its g.l.b. (greatest lower bound) and its l.u.b. (least upper bound) at least once on $[a, b]$.
4. A function $f(x)$ is **uniformly continuous** on a set S and only if for each positive real number ε there exists a real number δ such that $|x - X| < \delta$ implies $|f(x) - f(X)| < \varepsilon$ for all X in S.

If a function is continuous in a bounded closed interval $[a, b]$, it is uniformly continuous on $[a, b]$.
If $f(x)$ and $g(x)$ are continuous at a point, so are the functions $f(x) + g(x)$ and $f(x)\,f(x)$.

Limits

1. A function $f(x)$ of a real variable x has the right-hand limit $\lim_{x\to a+} f(x) = f(a+) = L_+$ at $x = a$ if and only if for each positive real number ε there exists a real number δ such that $0 < x - a < \delta$ implies that $f(x)$ is defined and $|f(x) - L_+| < \varepsilon$.
2. A function $f(x)$ of a real variable x has the **left-hand limit** $\lim_{x\to a-} f(x) = f(a-) = L_-$ at $x = a$ if and only if for each positive real number ε there exists a real number δ such that $0 < a - x < \delta$ implies that $f(x)$ is defined and $|f(x) - L_-| < \varepsilon$.
3. If $\lim_{x\to a} f(x)$ exists, then $\lim_{x\to a+} f(x) = \lim_{x\to a-} f(x) = \lim_{x\to a} f(x)$. Consequently, $\lim_{x\to a+} f(x) = \lim_{x\to a-} f(x)$ implies the existence of $\lim_{x\to a} f(x)$.
4. The function $f(x)$ is **right continuous** at $x = a$ if $f(a+) = f(a)$.
5. The function $f(x)$ is **left continuous** at $x = a$ if $f(a-) = f(a)$.
6. A real function $f(x)$ has a **discontinuity of the first kind** at point $x = a$ if $f(a+)$ and $f(a-)$ exist. The greatest difference between two of these numbers $f(a)$, $f(a+)$, $f(a-)$ is the **saltus** of $f(x)$ at the discontinuity. The discontinuities of the first kind of $f(x)$ constitute a discrete and countable set.
7. A real function $f(x)$ is **piecewise continuous** in an interval I if and only if $f(x)$ is continuous throughout I except for a finite number of discontinuities of the first kind.

Monotonicity

1. A real function $f(x)$ of a real variable x is a **strongly monotonic** in the open interval (a, b) if $f(x)$ increases as x increases in (a, b) or if $f(x)$ decreases as x decreases in (a, b).
2. A function $f(x)$ is **weakly monotonic** in (a, b) if $f(x)$ does not decrease, or if $f(x)$ does not increase in (a, b). Analogous definitions apply to monotonic sequences.
3. A real function of a real variable x is of **bounded variation** in the interval (a, b) if and only if there exists a real number of M such that

$$\sum_{i=1}^{m} |f(x_i) - f(x_{i-1})| < M \quad \text{for all partitions}$$

$$a = x_0 < x_1 < x_2 < \cdots < x_m = b$$

of the interval (a, b). If $f(x)$ and $g(x)$ are of bounded variation in (a, b), then $f(x) + g(x)$ and $f(x)g(x)$ are of bounded variation also. The function $f(x)$ is of bounded variation in every finite open interval where $f(x)$ is bounded and has a finite number of relative maxima and minima and discontinuities (Dirichlet conditions).

A function of bounded variation in (a, b) is bounded in (a, b) and its discontinuities are only of the first kind.

Table 1.2.2 presents some useful mathematical functions.

1.1.3 Energy and Power Signals

Energy Signals

If we consider any signal $f(t)$ as denoting a voltage that exists across a 1-ohm resistor, then

$$\frac{f^2(t)}{1} = f(t)\frac{f(t)}{1} = f(t)i(t) = \text{power } VA$$

Therefore, the integral

$$E = \int_a^b f^2(t)dt \quad \text{joule} \tag{1.3.1}$$

representing the energy dissipated in the resistor during the time interval (a, b). A signal is called **energy signal** if

$$\int_{-\infty}^{\infty} f^2(t)dt < \infty \tag{1.3.2}$$

Power Signals

Power signals are defined by the relation

$$0 \le \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f^2(t)dt < \infty \tag{1.3.3}$$

For complex-valued signals, we must introduce $|f(t)|^2$ instead of $f^2(t)$.

We may represent the energy in a finite interval in terms of the coefficients of the basis function φ_i; that is, we write the energy integral in the form

$$E = \int_a^b f^2(t)dt = \int_a^b f(t)\sum_{n=0}^{\infty} c_n\varphi_n(t)dt = \sum_{n=0}^{\infty} c_n\int_a^b f(t)\varphi_n(t)dt = \sum_{n=0}^{\infty} c_n^2\left\|\varphi_n(t)\right\|^2 \tag{1.3.4}$$

where

$$\int_a^b f(t)\varphi_n(t)dt = c_n\int_a^b \varphi_n^2(t)dt = c_n\left\|\varphi_n(t)\right\|^2$$

Because the square of the norm $\|\varphi_n(t)\|^2$ is the energy associated with the nth orthogonal function, (1.3.4) shows that the energy of the signal is the sum of the energies of its individual orthogonal components

TABLE 1.2.1 Some Useful Mathematical Functions

1. Signum Function

$$\operatorname{sgn}(t) = \begin{cases} 1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0 \end{cases}$$

2. Step Function

$$u(t) = \frac{1}{2} + \frac{1}{2}\operatorname{sgn}(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$

3. Ramp Function

$$r(t) = \int_{-\infty}^{t} u(\tau)\,d\tau = t u(t)$$

4. Pulse Function

$$p_a(t) = u(t + a) - u(t - a) = \begin{cases} 1 & |t| < a \\ 0 & |t| > a \end{cases}$$

5. Triangular Pulse

$$\Lambda_a(t) = \begin{cases} 1 - \frac{|t|}{a} & |t| < a \\ 0 & |t| > a \end{cases}$$

6. Sinc Function

$$\operatorname{sinc}_a(t) = \frac{\sin at}{t}, \qquad -\infty < t < \infty$$

7. Gaussian Function

$$g_a(t) = e^{-at^2}, \qquad -\infty < t < \infty$$

8. Error Function

$$\operatorname{erf}(t) = \frac{2}{\sqrt{\pi}}\int_0^t e^{-\tau^2}\,d\tau = \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n t^{2n+1}}{n!(2n+1)}$$

Properties:

$$\operatorname{erf}(\infty) = 1,\ \operatorname{erf}(0) = 0,\ \operatorname{erf}(-t) = -\operatorname{erf}(t)$$

$$\operatorname{erfc}(t) = \text{complementary error function} = 1 - \operatorname{erf}(t) = \frac{2}{\sqrt{\pi}}\int_t^{\infty} e^{-\tau^2}\,d\tau$$

9. Exponential Function

$$f(t) = e^{-at}u(t), \qquad t \ge 0$$

10. Double Exponential

$$f(t) = e^{-a|t|}, \qquad -\infty < t < \infty$$

11. Lognormal Function

$$f(t) = \frac{1}{t}e^{-(\ln t)^2/2}, \qquad 0 < t < \infty$$

12. Rayleigh Function

$$f(t) = te^{-t^2/2}, \qquad 0 \le t < \infty$$

weighted by c_n. Note that this is the Parseval theorem. This equation shows that the set $\{\varphi_n(t)\}$ forms an orthogonal (complete) set, and the signal energy can be calculated from this representation.

Example

(a) $\int_{-\infty}^{\infty} u^2(t)\,dt = \int_0^{\infty} dt = \infty$; $\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} u^2(t)\,dt = \lim_{T\to\infty} \frac{1}{2T}\int_0^{T} dt = \lim_{T\to\infty} \frac{1}{2T}(T) = \frac{1}{2} < \infty$.

This implies that $u(t)$ is a power signal.

(b) The signal $e^{-at}u(t)$, $a > 0$ is an energy signal.

1.2 Distributions, Delta Function

1.2.1 Introduction

The delta function $\delta(t)$ often called the **impulse** or **Dirac delta** function, occupies a central place in signal analysis. Many physical phenomena such as point sources, point charges, concentrated loads on structures, and voltage or current sources, acting for very short times, can be modeled as delta functions.

Strictly speaking, delta functions are not functions in the accepted mathematical sense, and they cannot be treated with rigor within the framework of classical analysis. However, if distributions are introduced, then the concept of a delta function and operations on delta functions can be given a precise meaning.

1.2.2 Testing Functions

A **distribution** is a generalization of a function. Within the framework of distributions, any function encountered in applications, such as unit step functions and pulses, may be differentiated as many times as we desire, and any convergent series of functions may be differentiated term by term.

A **testing function** $\varphi(t)$ is a real-valued function of the real variable that can be differentiated an arbitrary number of times, and which is identical to zero outside a finite interval.

Example
Testing function

$$\varphi(t,a)=\begin{cases} e^{-\frac{a^2}{a^2-t^2}} & |t|<a \\ 0 & |t|\geq a \end{cases} \tag{2.2.1}$$

Properties

1. If $f(t)$ can be differentiated arbitrarily often

$$\psi(t)=f(t)\,\varphi(t)=\text{testing function}$$

2. If $f(t)$ is zero outside a finite interval

$$\psi(t)=\int_{-\infty}^{\infty} f(\tau)\,\varphi(t-\tau)\,d\tau, \quad -\infty<t<\infty=\text{testing function}$$

3. A sequence of testing functions, $\{\varphi_n\}$, $1 \leq n < \infty$, converges to zero if all φ_n are identically zero outside some interval independent of n and each φ_n, as well as all of its derivatives, tends uniformly to zero.

Example:

$$\varphi_n(t)=\varphi\left(t+\frac{1}{n}\right)\to\varphi(t)$$

4. Testing functions belong to a set D, where D is a linear vector space, and if $\varphi_1 \in D$ and $\varphi_2 \in D$, then $\varphi_1+\varphi_2 \in D$ and $\alpha\varphi_1 \in D$ for any number α.

1.2.3 Definition of Distributions

A **distribution** (or generalized function) $g(t)$ is a process of assigning to an arbitrary test function $\varphi(t)$ a number $N_g[\varphi(t)]$. A distribution is also a functional.

Example
An ordinary function $f(t)$ is a distribution if

$$\int_{-\infty}^{\infty} f(t)\,\varphi(t)\,dt = N_f[\varphi(t)] \tag{2.3.1}$$

exists for every test function $\varphi(t)$ in the set. For example, if $f(t) = u(t)$ then

$$\int_{-\infty}^{\infty} u(t)\,\varphi(t)\,dt = \int_0^{\infty}\varphi(t)\,dt \tag{2.3.2}$$

The function $u(t)$ is a distribution that assigns to $\varphi(t)$ a number equal to its area from zero to infinity.

Properties of Distributions

1. Linearity–Homogeneity

$$\int_{-\infty}^{\infty} g(t)\left[a_1\varphi_1(t)+a_2\varphi_2(t)\right]dt = a_1\int_{-\infty}^{\infty} g(t)\,\varphi_1(t)\,dt + a_2\int_{-\infty}^{\infty} g(t)\,\varphi_2(t)\,dt \tag{2.3.3}$$

   for all test functions and all numbers a_i.

2. Summation

$$\int_{-\infty}^{\infty}\left[g_1(t)+g_2(t)\right]\varphi(t)\,dt = \int_{-\infty}^{\infty} g_1(t)\,\varphi(t)\,dt + \int_{-\infty}^{\infty} g_2(t)\,\varphi(t)\,dt \tag{2.3.4}$$

3. Shifting

$$\int_{-\infty}^{\infty} g(t-t_0)\,\varphi(t)\,dt = \int_{-\infty}^{\infty} g(t)\,\varphi(t+t_0)\,dt \tag{2.3.5}$$

4. Scaling

$$\int_{-\infty}^{\infty} g(at)\,\varphi(t)\,dt = \frac{1}{|a|}\int_{-\infty}^{\infty} g(t)\,\varphi\!\left(\frac{t}{a}\right)dt \tag{2.3.6}$$

5. Even Distribution

$$\int_{-\infty}^{\infty} g(t)\,\varphi(t)\,dt = 0, \quad \varphi(t) = \text{odd} \tag{2.3.7}$$

6. Odd Distribution

$$\int_{-\infty}^{\infty} g(t)\,\varphi(t)\,dt = 0, \quad \varphi(t) = \text{even} \tag{2.3.8}$$

7. Derivative

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{dg(t)}{dt}\,\varphi(t)\,dt &= g(t)\,\varphi(t)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} g(t)\,\frac{d\varphi(t)}{dt}\,dt \\ &= -\int_{-\infty}^{\infty} g(t)\,\frac{d\varphi(t)}{dt}\,dt\end{aligned} \tag{2.3.9}$$

   where the integrated term is equal to zero in view of the properties of testing functions.

8. The nth Derivative

$$\int_{-\infty}^{\infty} \frac{d^n g(t)}{dt^n}\varphi(t)dt = (-1)^n \int_{-\infty}^{\infty} g(t)\frac{d^n \varphi(t)}{dt^n}dt \tag{2.3.10}$$

9. Product with Ordinary Function

$$\int_{-\infty}^{\infty} [g(t)f(t)]\varphi(t)dt = \int_{-\infty}^{\infty} g(t)[f(t)\varphi(t)]dt \tag{2.3.11}$$

provided that $f(t)\varphi(t)$ belongs to the set of test functions.

10. Convolution

$$\begin{aligned}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} g_1(\tau)g_2(t-\tau)d\tau\right]\varphi(t)dt \\ = \int_{-\infty}^{\infty} g_1(\tau)\left[\int_{-\infty}^{\infty} g_2(t-\tau)\varphi(t)dt\right]d\tau\end{aligned} \tag{2.3.12}$$

by formal change of the order of integration.

Definition

A sequence of distributions $\{g_n(t)\}$ is said to converge to the distribution $g(t)$ if

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} g_n(t)\varphi(t)dt = \int_{-\infty}^{\infty} g(t)\varphi(t)dt \tag{2.3.13}$$

for all φ belonging to the set of test functions.

11. Every distribution is the limit, in the sense of distributions, of a sequence of infinitely differentiable functions.
12. If $g_n(t) \to g(t)$ and $r_n(t) \to r(t)$ (r is a distribution), and the numbers $a_n \to a$, then

$$\frac{d}{dt}g_n(t) \to \frac{dg(t)}{dt}, \quad g_n(t)+r_n(t) \to g(t)+r(t), \quad a_n g_n(t) \to a g(t) \tag{2.3.14}$$

13. Any distribution $g(t)$ may be differentiated as many times as desired. That is, the derivative of any distribution always exists and it is a distribution.

1.2.4 The Delta Function

Properties

Based on the distribution properties, the properties of the delta function are given below.

1. The delta function is a distribution assigning to the function $\varphi(t)$ the number $\varphi(0)$; thus

$$\int_{-\infty}^{\infty} \delta(t)\varphi(t)dt = \varphi(0) \tag{2.4.1}$$

2. Shifted

$$\int_{-\infty}^{\infty} \delta(t-t_0)\varphi(t)dt = \varphi(t_0) \tag{2.4.2}$$

3. Scales.

$$\int_{-\infty}^{\infty} \delta(at)\varphi(t)dt = \frac{1}{|a|}\int_{-\infty}^{\infty} \delta(t)\,\varphi\left(\frac{t}{a}\right)dt = \frac{1}{|a|}\varphi(0)$$

From (2.4.1) we have the identity

$$\delta(at) = \frac{1}{|a|}\delta(t)$$

and hence ($a = -1$)

$$\delta(-t) = \delta(t) = \text{even} \tag{2.4.3}$$

4. Multiplication by Continuous Function

$$\int_{-\infty}^{\infty}[\delta(t)f(t)]\varphi(t)dt = \int_{-\infty}^{\infty}\delta(t)[f(t)\varphi(t)]dt = f(0)\varphi(0)$$

If $f(t)$ is continuous at 0, then

$$f(t)\delta(t) = f(0)\delta(t) \tag{2.4.4}$$

and

$$t\delta(t) = 0 \tag{2.4.5}$$

5. Derivatives

$$\int_{-\infty}^{\infty}\frac{d\delta(t)}{dt}\varphi(t)dt = -\frac{d\varphi(0)}{dt}$$

$$\int_{-\infty}^{\infty}\frac{d\delta(t-t_0)}{dt}\varphi(t)dt = -\frac{d\varphi(t_0)}{dt} \tag{2.4.6}$$

$$\int_{-\infty}^{\infty}\frac{d^n\delta(t)}{dt^n}\varphi(t)dt = (-1)^n\frac{d^n\varphi(0)}{dt^n} \tag{2.4.7}$$

$$\int_{-\infty}^{\infty}\frac{d\delta(t)}{dt}f(t)\varphi(t)dt = -\int_{-\infty}^{\infty}\delta(t)\frac{d[f(t)\varphi(t)]}{dt}dt$$

$$= -f(0)\frac{d\varphi(0)}{dt} - \frac{df(0)}{dt}\varphi(0) \tag{2.4.8}$$

$$f(t)\frac{d\delta(t)}{dt} = -\frac{df(0)}{dt}\delta(t) + f(0)\frac{d\delta(t)}{dt} \tag{2.4.9}$$

$$t\frac{d\delta(t)}{dt} = -\delta(t) \tag{2.4.10}$$

Set $f(t) = \varphi(t) = 1$ in (2.4.8) to find the relation

$$\int_{-\infty}^{\infty} \frac{d\delta(t)}{dt}\,dt = 0 \qquad \left[\frac{d\delta(t)}{dt} \text{ is an odd function}\right] \tag{2.4.11}$$

$$f(t)\frac{d^n\delta(t)}{dt^n} = \sum_{k=0}^{n}(-1)^k \frac{n!}{k!(n-k)!}\frac{d^k f(0)}{dt^k}\frac{d^{n-k}\delta(t)}{dt^{n-k}} \tag{2.4.12}$$

From

$$\int_{-\infty}^{\infty} \frac{du(t)}{dt}\varphi(t)\,dt = u(t)\varphi(t)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} u(t)\frac{d\varphi(t)}{dt}\,dt$$

$$= -\int_{0}^{\infty} \frac{d\varphi(t)}{dt}\,dt = -\varphi(t)\Big|_{0}^{\infty} = \varphi(0)$$

and comparing with (2.1.1) we find that

$$\delta(t) = \frac{du(t)}{dt} \tag{2.4.13}$$

Therefore, the generalized derivatives of discontinuous function contain impulses. A_n is the jump at the discontinuity point $t = t_n$ of the expression $A_n\,\varphi(t - t_n)$. Also

$$\frac{d\delta(t)}{dt} = \frac{d^2u(t)}{dt^2} \quad \text{or} \quad u(t) + u(-t) = 1$$

Hence

$$\frac{du(-t)}{dt} = -\delta(t) \tag{2.4.14}$$

$$\delta(t - t_0) = \frac{du(t - t_0)}{dt} \tag{2.4.15}$$

If $r(t)$ has a finite or countably infinite number of zeros at t_n on the entire t axis and these points $r(t)$ have a continuous derivative $dr(t_n)/dt \neq 0$, then

$$\delta[r(t)] = \sum_{n} \frac{\delta(t - t_n)}{\left|\frac{dr(t_n)}{dt}\right|} \tag{2.4.16}$$

Hence, we obtain

$$\delta(t^2 - 1) = \frac{1}{2}\delta(t - 1) + \frac{1}{2}\delta(t + 1) \tag{2.4.17}$$

[illegible]

$$\delta(\sin t) = \sum_{n=-\infty}^{\infty} \delta(t - n\pi) \tag{2.4.18}$$

In addition, the following relation is also true:

$$\frac{d\delta[r(t)]}{dt} = \sum_n \frac{\dfrac{d\delta(t - t_n)}{dt}}{\dfrac{dr(t)}{dt}\left|\dfrac{dr(t_n)}{dt}\right|} \tag{2.4.19}$$

6. Integrals

$$\int_{-\infty}^{\infty} A\delta(t - t_0)\,dt = A \tag{2.4.20}$$

for all t_0

$$\begin{aligned} \delta(t - t_1) * \delta(t - t_2) &= \text{convolution} \\ &= \int_{-\infty}^{\infty} \delta(\tau - t_1)\delta(t - \tau - t_2)\,d\tau = \delta[t - (t_1 + t_2)] \end{aligned} \tag{2.4.21}$$

$$f(t) * \delta(t) = \int_{-\infty}^{\infty} f(t - \tau)\delta(\tau)\,d\tau = f(t - 0) = f(t) \tag{2.4.22}$$

Distributions as Generalized Limits

We can define a distribution as a generalized limit of a sequence $f_n(t)$ of ordinary function. If there exists a sequence $f_n(t)$ such that the limit

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(t)\varphi(t)\,dt \tag{2.4.23}$$

exists for every test function in the set, then the result is a number depending on $\varphi(t)$. Hence, we may define a distribution $g(t)$ as

$$g(t) = \lim f_n(t) \tag{2.4.24}$$

and, therefore, equivalently

$$\delta(t) = \lim f_n(t) \tag{2.4.25}$$

Consider the two sequences shown in Figures 2.4.1a and 2.4.1b. The rectangular pulse sequence is given by

$$p_\varepsilon(t) = \frac{u(t) - u(t - \varepsilon)}{\varepsilon}$$

and has area unity whatever the value of ε. Because $\varphi(t)$ is continuous, it follows that sample content of The Transforms and Applications Handbook (2nd Edition) (The Electronic Engineering Handbook Series)