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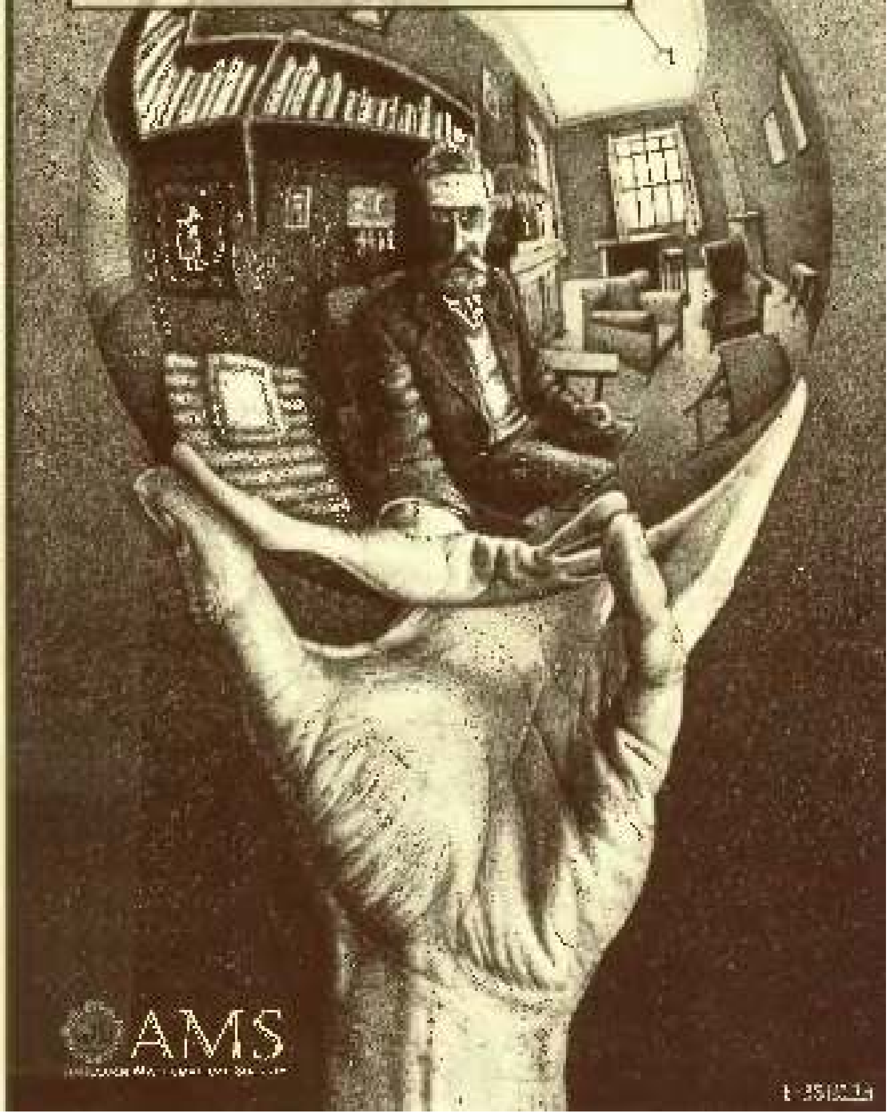
Inversion Theory and Conformal Mapping

David E. Blair



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2000 *Mathematics Subject Classification.* Primary 26B10, 30–01, 30C99, 51–01, 51B10, 51M04, 52A10, 53A99.

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About the cover: The picture on the cover is M. C. Escher's "Hand with Reflecting Sphere", a self-portrait of the artist. Reflection (inversion) in a sphere is a conformal map; thus, while distances are distorted, angles are preserved and the image is recognizable. It is for this reason that we commonly use spherical mirrors, e.g. the right hand rear view mirror on an automobile.

Library of Congress Cataloging-in-Publication Data

Blair, David E., 1940–
Inversion theory and conformal mapping / David E. Blair.
p. cm. — (Student mathematical library, ISSN 1520-9121 ; v. 9)
Includes bibliographical references and index.
ISBN 0-8218-2636-0 (softcover : alk. paper)
1. Inversions (Geometry) 2. Conformal mapping. I. Title. II. Series.
QA473.B57 2000
516.3′6—dc21 00-033137

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To Marie and Matthew

Contents

Preface

It is rarely taught in an undergraduate, or even graduate, curriculum that the only conformal maps in Euclidean space of dimension greater than 2 are those generated by similarities and inversions (reflections) in spheres. This contrasts with the abundance of conformal maps in the plane, a fact which is taught in most complex analysis courses. The principal aim of this text is to give a treatment of this paucity of conformal maps in higher dimensions. The result was proved in 1850 in dimension 3 by J. Liouville [**22**]. In Chapter 5 of the present text we give a proof in general dimension due to R. Nevanlinna [**26**] and in Chapter 6 give a differential geometric proof in dimension 3 which is often regarded as the classical proof, though it is not Liouville's proof. For completeness, in Chapter 4 we develop enough complex analysis to prove the abundance of conformal maps in the plane.

In addition this book develops inversion theory as a subject along with the auxiliary theme of "circle preserving maps".

The text as presented here is at the advanced undergraduate level and is suitable for a "capstone course", topics course, senior seminar, independent study, etc. The author has successfully used this material for capstone courses at Michigan State University. One particular feature is the inclusion of the paper on circle preserving transformations by C. Carathéodory [**6**]. This paper divides itself up nicely into small sections, and students were asked to present the paper to the class. This turned out to be an enjoyable and profitable experience for the students. When there were more than enough students in the class for this exercise, some of the students presented Section 2.8.

The author expresses his appreciation to Dr. Edward Dunne and the production staff of the American Mathematical Society for their kind assistance in producing this book.

Chapter 1

Classical Inversion Theory in the Plane

1.1. Definition and basic properties

Let C be a circle centered at a point O with radius r. If P is any point other than O, the *inverse* of P with respect to C is the point P' on the ray $\overrightarrow{OP}$ such that the product of the distances of P and P' from O is equal to r^2. Inversion in a circle is sometimes referred to as "reflection" in a circle; some reasons for this will become apparent as we progress.

Clearly if P' is the inverse of P, then P is the inverse of P'. Note also that if P is in the interior of C, P' is exterior to C, and vice-versa. So the interior of C except for O is mapped to the exterior and the exterior to the interior. C itself is left pointwise fixed. O has no image, and no point of the plane is mapped to O. However, points close to O are mapped to points far from O and points far from O map to points close to O. Thus adjoining one "ideal point", or "point at infinity", to the Euclidean plane, we can include O in the domain and range of inversion. We will treat this point at infinity in detail in Section 2.2.

We denote by $\overline{PQ}$ the length of the line segment PQ. The similarity and congruence of triangles will be denoted by $\sim$ and $\cong$ respectively.

Given C, note the ease with which we can construct the inverse of a point P. If P is interior to C, construct the perpendicular to OP at P meeting the circle at T; the tangent to C at T then meets $\overrightarrow{OP}$ at the inverse point P' (Figure 1.1). To see this, simply observe that $\triangle OPT \sim \triangle OTP'$ and hence

$$\frac{\overline{OP}}{\overline{OT}} = \frac{\overline{OT}}{\overline{OP'}}.$$

Therefore $\overline{OP} \cdot \overline{OP'} = \overline{OT}^2 = r^2$. If P is exterior to C, construct a tangent to C from P meeting C at T; the perpendicular from T to $\overline{OP}$ meets $\overrightarrow{OP}$ at the inverse point by virtue of the same argument.

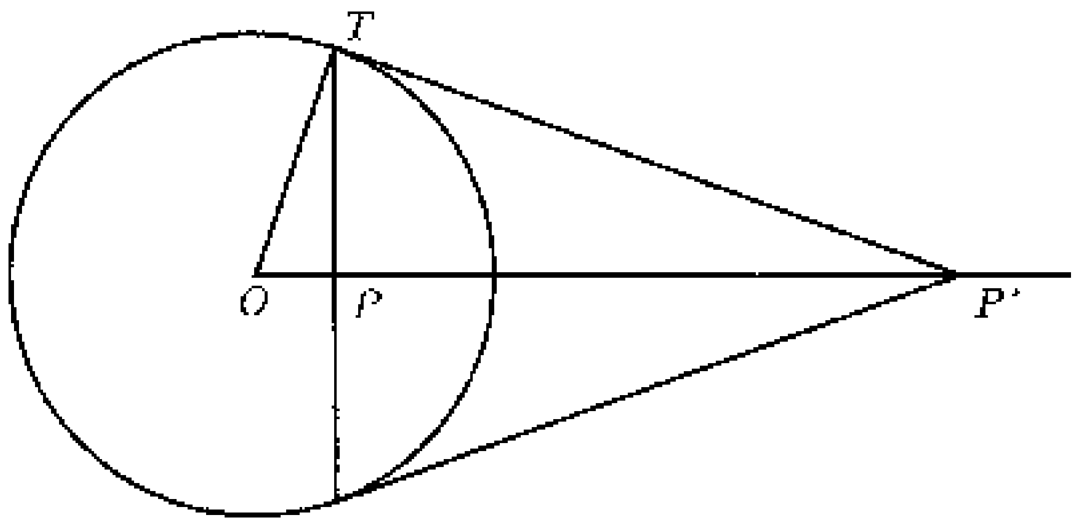

Figure 1.1

A common alternate construction of the inverse point is the following. Construct the diameter of C perpendicular to OP at O meeting C at points N and S. Draw $\overleftrightarrow{NP}$ meeting C at Q and draw $\overleftrightarrow{SQ}$ meeting $\overrightarrow{OP}$ at P' (Figure 1.2). Then $\triangle NOP \sim \triangle NQS \sim \triangle P'OS$ and hence $\overline{OP}/\overline{ON} = \overline{OS}/\overline{OP'}$, giving $\overline{OP} \cdot \overline{OP'} = \overline{ON} \cdot \overline{OS} = r^2$. Therefore P' is the inverse of P. Here P may be interior or exterior to C.

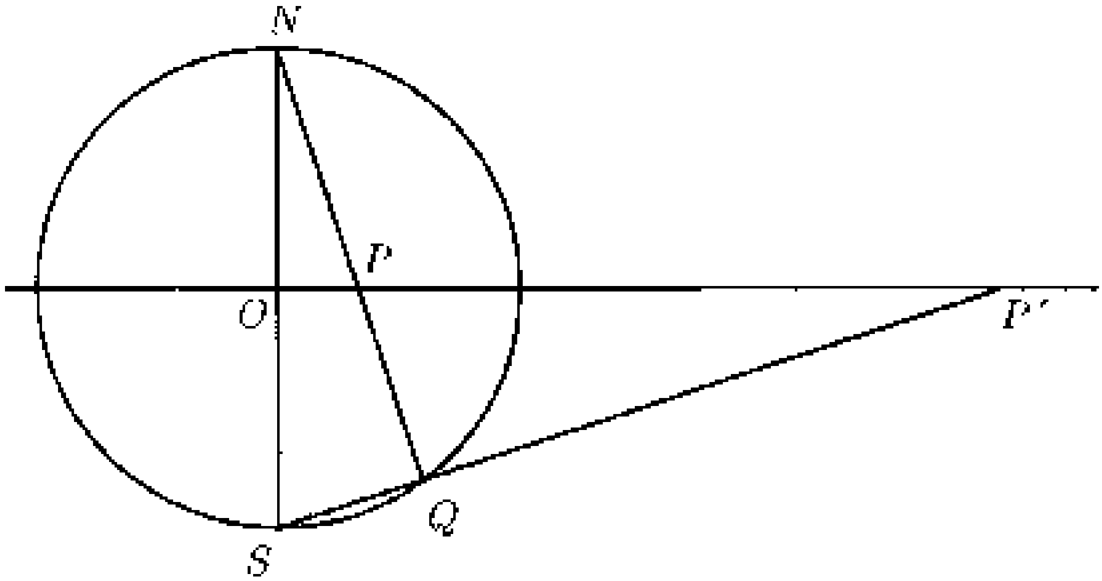

Figure 1.2

Yet another construction of the inverse of a point is given by the following exercise; though more complicated, we will make use of this construction in Chapter 2.

EXERCISE

Construct a radius of C perpendicular to OP at O meeting the circle at N, and construct the circle $\mathcal{D}$ with diameter ON. Draw NP meeting $\mathcal{D}$ at Q. Draw the parallel to ON through Q meeting $\mathcal{D}$ at Q'. Show that the ray $\overrightarrow{NQ'}$ meets $\overrightarrow{OP}$ at the inverse point P'.

The first basic property of inversion that we will prove is that lines and circles as a class are mapped to lines and circles.

Theorem 1.1. a) *The inverse of a line through the center of inversion is the line itself.*

b) *The inverse of a line not passing through the center of inversion is a circle passing through the center of inversion.*

c) *The inverse of a circle through the center of inversion is a line not passing through the center of inversion.*

d) *The inverse of a circle not passing through the center of inversion is a circle not passing through the center of inversion.*

Proof. Let C be the circle of inversion with center O and radius r. Since O is collinear with any pair of inverse points, a) is clear. For b), drop the perpendicular from O to the line meeting at P and let P' be the inverse of P (Figure 1.3).

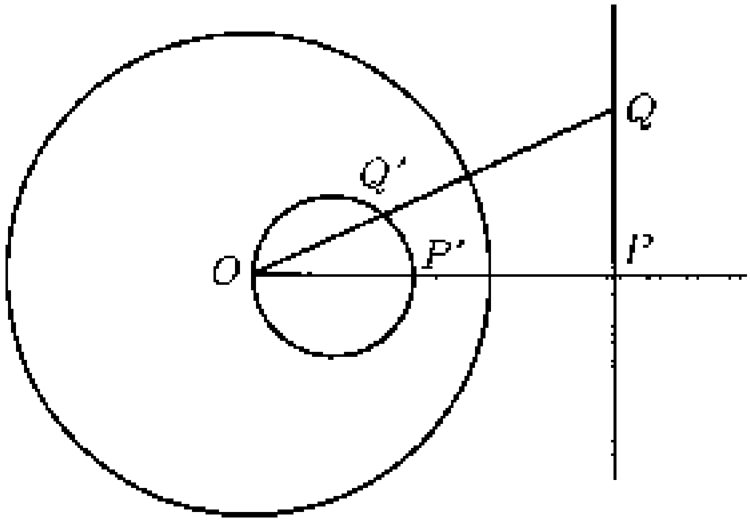

Figure 1.3

Let Q be any other point on the line and Q' its inverse. Then $\overline{OP} \cdot \overline{OP'} = \overline{OQ} \cdot \overline{OQ'} = r^2$ or $\overline{OP}/\overline{OQ} = \overline{OQ'}/\overline{OP'}$, and hence $\triangle OPQ \sim \triangle OQ'P'$. Therefore $\angle OQ'P'$ is a right angle, and hence Q' is on the circle $\mathcal{A}$ of diameter OP'. Thus the image of the line lies in the point set of $\mathcal{A}$; now reverse the argument to show that any point $Q' \neq O$ on $\mathcal{A}$ is the image of some point on the line.

To prove c), let P be the point on the given circle diametrically opposite to O and extend, if necessary, this diameter to the inverse point P' of P (in Figure 1.3 reverse the roles of P and P' and Q and Q'). Let Q be any other point on the circle and Q' its inverse. Again $\triangle OPQ \sim \triangle OQ'P'$. Therefore $\angle OP'Q'$ is a right angle and hence Q' is on the perpendicular to $\overrightarrow{OP}$ at P'; the result then follows as before.

Finally to prove d), let $\mathcal{A}$ be the given circle with center A. If $O = A$ the result is immediate, so assume $O \neq A$. Draw the line through O and A cutting $\mathcal{A}$ at P and Q, and let P' and Q' be the inverse points of P and Q respectively. Let R be any other point on $\mathcal{A}$, and R' its inverse (Figure 1.4). Then $\overline{OP} \cdot \overline{OP'} = \overline{OR} \cdot \overline{OR'} = r^2$ and hence $\triangle OPR \sim \triangle OR'P'$. Similarly $\triangle OQR \sim \triangle OR'Q'$. Thus $\angle OPR \cong \angle OR'P'$ and $\angle OQR \cong \angle OR'Q'$, but $\angle PRQ$ is a right angle and therefore $\angle P'R'Q'$ is a right angle. Thus as the point R moves on $\mathcal{A}$, R' moves on the circle $\mathcal{A}'$ with diameter $P'Q'$ and any point on $\mathcal{A}'$ is the image of a point of $\mathcal{A}$. ❐

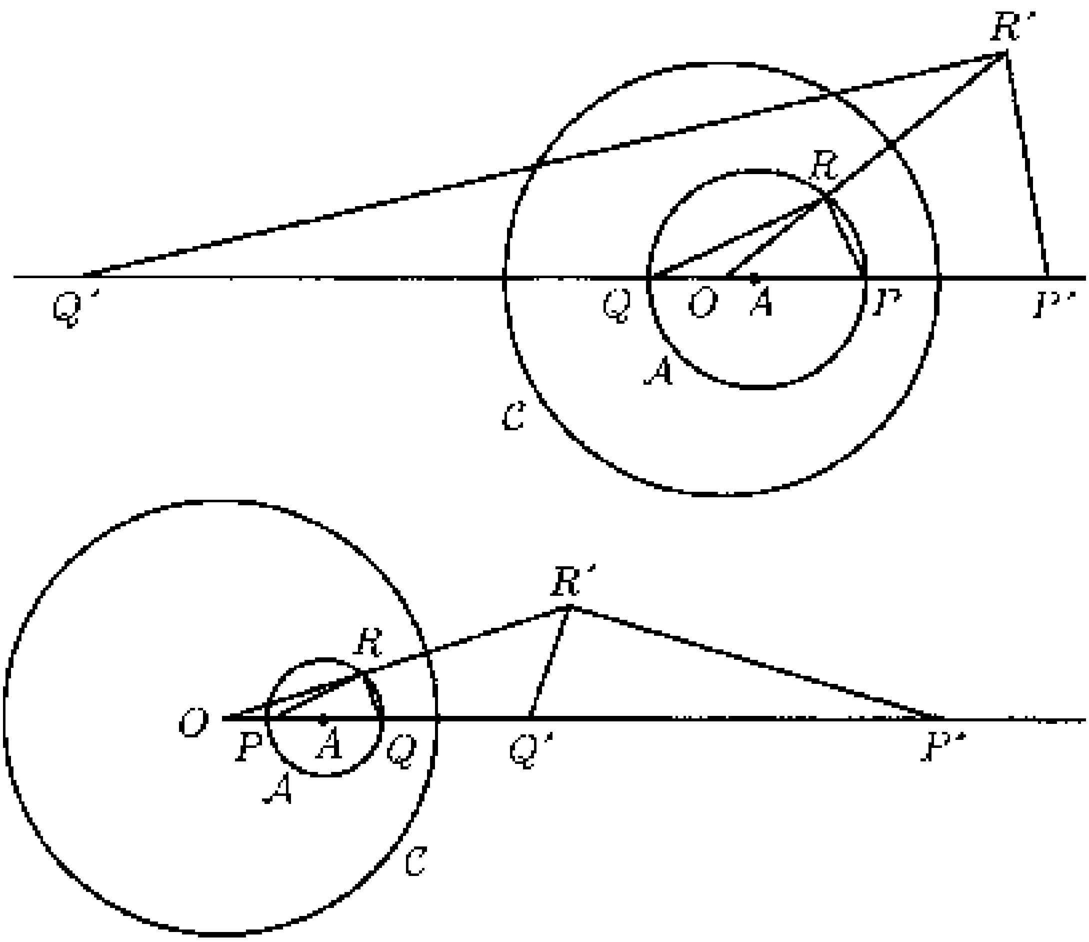

Figure 1.4

The second basic property of inversion is that a circle orthogonal to the circle of inversion inverts to itself.

Theorem 1.2. *Any circle through a pair of inverse points is orthogonal to the circle of inversion; and, conversely, any circle cutting the circle of inversion orthogonally and passing through a point P, passes through its inverse P'.*

This theorem is an immediate consequence of the well known theorem in Euclidean geometry that a tangent to a circle from an external point is the mean proportional between the segments of any secant from the point. To see this, consider the segment PT of a tangent to a circle from an external point P making contact at T.

Let R and S be the intersection points of a secant from P (Figure 1.5). Then $\triangle PRT \sim \triangle PTS$, and hence $\overline{PR}/\overline{PT} = \overline{PT}/\overline{PS}$.

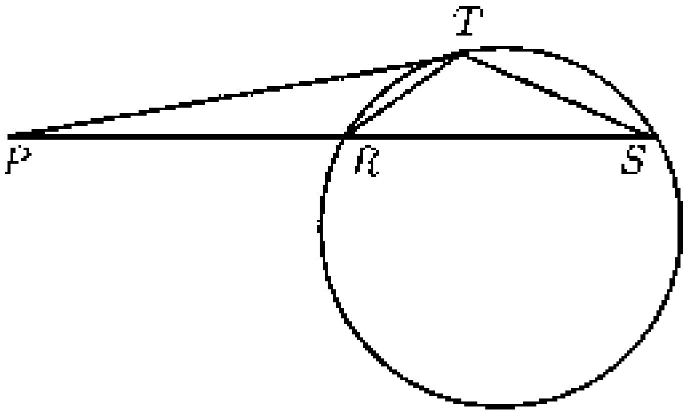

Figure 1.5

Corollary 1.1. *A circle orthogonal to the circle of inversion inverts to itself.*

Corollary 1.2. *Through two points P and Q in the interior of a circle C and not on the same diameter, there exists one and only one circle orthogonal to C.*

Remark. This last corollary is important for the Poincaré model of the hyperbolic plane. Consider a geometry whose points are the interior points of a circle C and whose lines are the diameters of C and arcs of circles orthogonal to C. The corollary assures the existence and uniqueness of a line through two given points. It is also easy to see that the parallel postulate of Euclidean geometry does not hold in this geometry. With some effort one can show that the other axioms of Euclidean geometry do hold in this geometry (see e.g. Greenberg [**16**]) and hence that the parallel postulate is independent of the other axioms of Euclidean geometry.

The third basic property of inversion that we consider is its conformality. Let C_1 and C_2 be two differentiable curves meeting at a point P with tangent lines at P. (Recall that if a plane curve is given parametrically by $x = x(t)$, $y = y(t)$ with not both $x'(t_0)$ and $y'(t_0)$ equal to zero, then the curve has a *tangent vector* or *velocity vector* at $(x(t_0), y(t_0))$, namely $x'(t_0)\mathbf{i} + y'(t_0)\mathbf{j}$ in classical vector notation.) By the *angle* between two curves we mean the undirected angle between their tangent vectors at P. A transformation T mapping a subset of the plane into the plane is said to be *conformal* at P if it preserves the angle between any two curves at P. T is said to be *conformal* if it is conformal at each point of its domain. Some authors require that the sense of angle be preserved as well as the magnitude, but here we define conformality in the wider sense; in fact, inversion reverses the sense of angles.

Our proof of conformality will use a formula found in many calculus texts, but since it is often omitted in first year courses, we briefly derive it here. Let (ρ, θ) be polar coordinates in the plane, and consider a differentiable curve $\rho = f(\theta)$. Let α be the angle of inclination of the tangent lines and $\psi = \alpha - \theta$ (Figure 1.6); then

$$\cot\psi = \frac{1}{\rho}\frac{d\rho}{d\theta} \text{ or } \frac{f'(\theta)}{f(\theta)}.$$

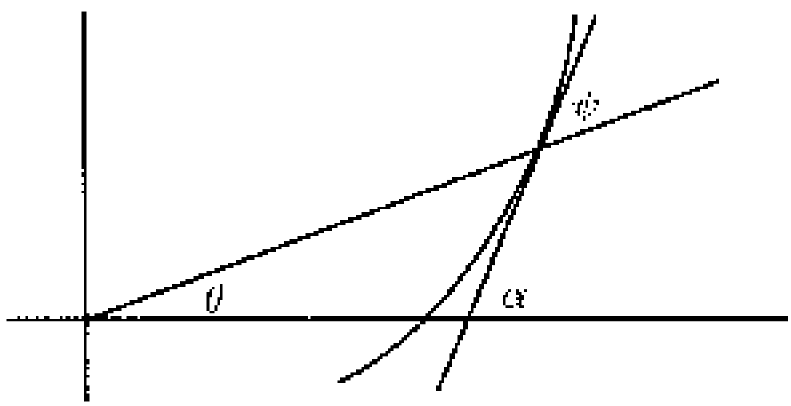

Figure 1.6

In cartesian coordinates the curve is given as $x = \rho\cos\theta$, $y = \rho\sin\theta$. Then

$$\tan\alpha = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{(d\rho/d\theta)\sin\theta + \rho\cos\theta}{(d\rho/d\theta)\cos\theta - \rho\sin\theta}.$$

Substituting this into

$$\tan\psi = \frac{\tan\alpha - \tan\theta}{1 + \tan\alpha\tan\theta}$$

and simplifying gives the desired formula.

Theorem 1.3. *Inversion in a circle is a conformal map.*

Proof. Let (ρ, θ) be polar coordinates in the plane with the origin at the center of inversion and let r be the radius of the circle of inversion. Suppose that $\rho = f_i(\theta)$, $i = 1, 2$, are two differentiable curves meeting at a point P. Let $\rho = g_i(\theta)$, $i = 1, 2$, be the images of the two curves under inversion; then $g_i(\theta) = \frac{r^2}{f_i(\theta)}$. Let ψ_i and ϕ_i denote the undirected angle between the ray corresponding to θ and tangent to $\rho = f_i(\theta)$ and $\rho = g_i(\theta)$ respectively. Let $\beta = \psi_2 - \psi_1$ and $\beta' = \phi_2 - \phi_1$ at P; we shall show that $\beta = \beta'$ to within sign. Since $g_i(\theta) = r^2/f_i(\theta)$, we get

$$g_i'(\theta) = -\frac{r^2 f_i'(\theta)}{(f_i(\theta))^2},$$

and hence

$$\cot\phi_i = \frac{g_i'(\theta)}{g_i(\theta)} = -\frac{f_i'(\theta)}{f_i(\theta)} = -\cot\psi_i.$$

Therefore

$$\cot\beta' = \frac{\cot\phi_2\cot\phi_1 - 1}{\cot\phi_1 - \cot\phi_2} = -\cot\beta = \cot(-\beta).$$

□

Exercises

1. Let O be a point on a circle with center C and suppose the inverse of this circle with respect to O as center of inversion intersects $\overrightarrow{OC}$ at the point A'. If C' is the inverse of C, show that $\overline{OA'} = \overline{A'C'}$.

2. Find the equation of the circle that is the inverse of the line $ax + by = c, c \neq 0$, under inversion in the circle $x^2 + y^2 = 1$.

3. Let P, P' and Q, Q' be two pairs of points, inverse with respect to a circle $\mathcal{C}$. Show that a circle passing through three of these points passes through the fourth.

4. Let C_1 and C_2 be two circles intersecting in two points P and Q. If C_1 and C_2 are both orthogonal to a third circle C_3 with center O, show that O, P and Q are collinear.

5. Given three collinear points O, P, P' with O not between P and P', construct a circle centered at O with respect to which P and P' are inverse points.

6. Let P and Q be inverse points with respect to a circle $\mathcal{A}$. Prove that inversion in a circle with center $O \neq P, Q$ nor on $\mathcal{A}$ maps P and Q to points P' and Q' which are inverse with respect to the image circle $\mathcal{A}'$. In particular, inversion is an inversive invariant. Hint: Consider two circles $\mathcal{B}$ and $\mathcal{D}$ passing through both P and Q but neither passing through O.

7. Discuss the meaning of Exercise 6 when O is on $\mathcal{A}$ and when O is the point P or Q.

8. Show that the inverse of the center of a circle $\mathcal{A}$ orthogonal to circle of inversion $\mathcal{C}$ is the midpoint of the common chord. More generally, show that the inverse of the center A of a circle $\mathcal{A}$ not through the center of inversion O is the inverse of O in the circle which is the inverse of $\mathcal{A}$. (If O is exterior to $\mathcal{A}$ this is fairly easy using a tangent from O to $\mathcal{A}$ and its inverse. For a clever proof, use Exercise 7. Another proof can be given using the ideas of the next section.)

9. Let $\mathcal{C}$ be the circle $x^2 + y^2 = 1$ in the xy-plane; find the equation of the circle through $(\frac{1}{2}, 0)$, $(0, \frac{1}{2})$ orthogonal to $\mathcal{C}$.

10. Given circle $\mathcal{C}$ with center O, point $P \neq O$ interior to $\mathcal{C}$, and line l through P but not through O, construct the circle through P, tangent to l and orthogonal to $\mathcal{C}$.

11. Compass Construction of the Inverse: Given a circle $\mathcal{C}$ with center O and point P exterior to $\mathcal{C}$, draw the circle centered at P and passing through O meeting $\mathcal{C}$ at R and S. Draw circles centered at R and S and passing through O; let P' be the other point of intersection of these circles. Prove that P' is the inverse of P in the circle $\mathcal{C}$.

1.2. Cross ratio

Let $\vec{AB}$ denote the *directed distance* from A to B along a line l; that is, we designate a positive direction or orientation on l, and $\vec{AB} = \overline{AB}$ if the ray with initial point A containing B has the positive direction of the orientation and $\vec{AB} = -\overline{AB}$ if the ray has the opposite direction. Clearly $\vec{AB} = -\vec{BA}$.

Lemma 1.1. *If A, B and C are collinear, then $\vec{AB}+\vec{BC}+\vec{CA}=0$.*

Proof. The proof is by cases. If C is between A and B, then $\vec{AB} = \vec{AC}+\vec{CB}$ or $\vec{AB}-\vec{AC}-\vec{CB}=0$. Therefore $\vec{AB}+\vec{BC}+\vec{CA}=0$. The proofs of the other cases are similar. □

Lemma 1.2. *Let AB be a segment of a line l and O any point of l. Then $\vec{AB}=\vec{OB}-\vec{OA}$.*

Proof. $\vec{AB}+\vec{BO}+\vec{OA}=0$ by Lemma 1.1, and hence $\vec{AB}=\vec{OB}-\vec{OA}$. □

Let AB be a segment of a line l and $P \in l$. P is said to *divide* AB in the ratio $\vec{AP}/\vec{PB}$. This ratio has several basic properties, which we now present.

(1) The ratio is independent of the orientation of l.

(2) The ratio is positive if P is between A and B, and negative if P is exterior to AB.

(3) If $\vec{AP}/\vec{PB}=\vec{AP'}/\vec{P'B}$, then $P=P'$.
Proof. We have

$$\frac{\vec{AP}+\vec{PB}}{\vec{PB}}=\frac{\vec{AP'}+\vec{P'B}}{\vec{P'B}},$$

so by Lemma 1.1

$$\frac{\vec{AB}}{\vec{PB}}=\frac{\vec{AB}}{\vec{P'B}}.$$

Therefore $\vec{PB}=\vec{P'B}$ or $\vec{BP}=\vec{BP'}$, and hence $P=P'$. □

(4) If $r \neq -1$, there exists a point P such that $\vec{AP}/\vec{PB}=r$.
Proof. Consider the equation

$$r=\frac{\vec{AP}}{\vec{AB}-\vec{AP}};$$

solving gives $\vec{AP}=\frac{r}{1+r}\vec{AB}$. Then, given $r\neq -1$, we can find the point P. □

(5) $\displaystyle\lim_{P\to\infty}\frac{\vec{AP}}{\vec{PB}}=-1.$

Proof. Indeed,

$$r = \frac{\vec{AP}}{\vec{AB} - \vec{AP}} = \frac{1}{\frac{\vec{AB}}{\vec{AP}} - 1} \longrightarrow -1.$$

□

Suppose now that A, B, C and D are four distinct points on an oriented line l; we define their *cross ratio* (AB, CD) by

$$(AB, CD) = \frac{\vec{AC}/\vec{CB}}{\vec{AD}/\vec{DB}}.$$

Note that the cross ratio is positive if both C and D are between A and B or if neither C nor D is between A and B, whereas the cross ratio is negative if the pairs $\{A, B\}$ and $\{C, D\}$ separate each other.

Given three distinct points A, B and C on l and a real number $\mu \neq 0, 1, -\frac{\vec{AC}}{\vec{CB}}$, let D be the unique point dividing the segment AB in the ratio

$$\frac{1}{\mu}\frac{\vec{AC}}{\vec{CB}};$$

thus there exists a unique fourth point D such that $(AB, CD) = \mu$.

We say that four points on a line, A, B, C and D form a *harmonic set* (Figure 1.7) if

$$(AB, CD) = -1.$$

We denote a harmonic set of points by $H(AB, CD)$ and we say that C and D are *harmonic conjugates* with respect to A and B.

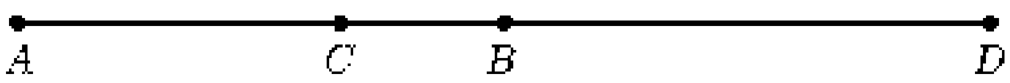

Figure 1.7

In Exercise 3 below one sees that $(AB, DC) = \frac{1}{(AB,CD)}$ and hence that the notion of harmonic conjugate is well defined: If D is the harmonic conjugate of C with respect to AB, C is the harmonic conjugate of D with respect to AB.

If $H(AB, CD)$, then the lengths of segments $\overline{AC}$, $\overline{AB}$, $\overline{AD}$ in this order form a harmonic progression. For if $(AB, CD) = -1$, then

$\frac{\vec{CB}}{\vec{AC}} = \frac{\vec{BD}}{\vec{AD}}$; but $\vec{CB} = \vec{AB} - \vec{AC}$ and $\vec{BD} = \vec{AD} - \vec{AB}$, and hence

$$\frac{\vec{AB} - \vec{AC}}{\vec{AB}\cdot\vec{AC}} = \frac{\vec{AD} - \vec{AB}}{\vec{AD}\cdot\vec{AB}}.$$

Thus

$$\frac{1}{\vec{AB}} = \frac{\frac{1}{\vec{AC}} + \frac{1}{\vec{AD}}}{2};$$

that is, $\frac{1}{\vec{AB}}$ is the arithemetic mean of $\frac{1}{\vec{AC}}$ and $\frac{1}{\vec{AD}}$.

Theorem 1.4. *Let C be a circle with center O, and C and D a pair of points inverse with respect to C. Let A and B be the endpoints of the diameter through C and D. Then $(AB, CD) = -1$. Conversely, if A and B are the endpoints of a diameter and $(AB, CD) = -1$, then C and D are inverse points.*

Proof. $\frac{\vec{AC}}{\vec{CB}} = -\frac{\vec{AD}}{\vec{DB}}$ is equivalent to

$$\frac{\vec{OC} - \vec{OA}}{\vec{OB} - \vec{OC}} = -\frac{\vec{OD} - \vec{OA}}{\vec{OB} - \vec{OD}};$$

but $\vec{OA} = -\vec{OB}$, so that

$$(\vec{OC} + \vec{OB})(\vec{OB} - \vec{OD}) = -(\vec{OD} + \vec{OB})(\vec{OB} - \vec{OC})$$

or $\vec{OC} \cdot \vec{OD} = \vec{OB}^2$. □

The following lemma is used here and in later applications.

Lemma 1.3. *Let C be a circle of inversion with center O and radius r. If P, P' and Q, Q' are pairs of inverse points, then*

$$P'Q' = r^2 \frac{\overline{PQ}}{\overline{OP}\cdot\overline{OQ}}.$$

Proof. We give the proof here in the case when O, P and Q are collinear; the non-collinear case is left to the reader in Exercise 1 below. $\vec{OP} \cdot \vec{OP'} = \vec{OQ} \cdot \vec{OQ'}$ but $\vec{OP} = \vec{OQ} + \vec{QP}$ and $\vec{OQ'} = \vec{OP'} + \vec{P'Q'}$, giving $\vec{QP} \cdot \vec{OP'} = \vec{P'Q'} \cdot \vec{OQ}$. Therefore

$$\overline{P'Q'} = \frac{\overline{QP}\cdot\overline{OP'}\cdot\overline{OP}}{\overline{OP}\cdot\overline{OQ}} = r^2 \frac{\overline{PQ}}{\overline{OP}\cdot\overline{OQ}}.$$

□

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