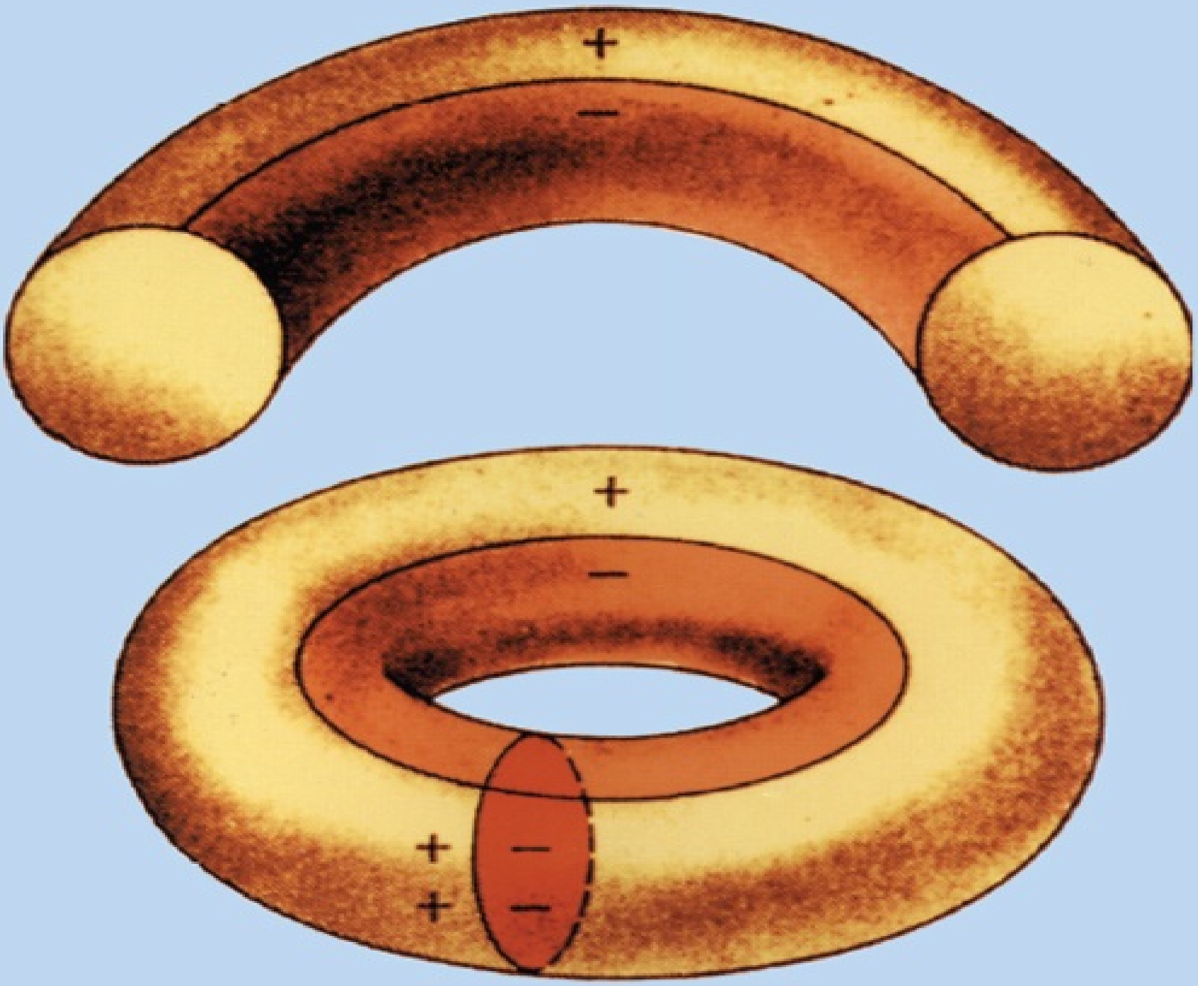
INTRODUCTION TO TOPOLOGY

Second Edition



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To our parents
Frank and Ruth Gamelin
Lee and Dorothy Greene

PREFACE

One of the most important developments in mathematics in the twentieth century has been t formation of topology as an independent field of study and the subsequent systematic application topological ideas to other fields of mathematics. Ideas that are topological in nature occurre explicitly in the nineteenth century and in embryonic form even earlier. The part of topology mo relevant to analysis, that part usually called point-set topology or general topology, had its beginnin in the nineteenth-century works that first established calculus on a rigorous basis. The beginnings the second large branch of topology, algebraic and geometric topology, are from an even earli period; for instance, Euler's results in the eighteenth century on the combinatorics of polyhedr figures were already a clear precursor of contemporary ideas. However, the systematic investigatio of topology as a separate field of mathematics, which began in the late nineteenth century a continues unabated today, has given the ideas of topology both new generality and new depth. By no the ideas of point-set topology are a large part of the basic language and technique of analysis. T methods of algebraic topology play an important role in algebra as well as forming a large field mathematics unto themselves. Also, the great growth of differential geometry in recent times associated with a viewpoint involving topological concepts.

The by now well-established importance of topology has led naturally to the writing of ma works on the subject, including a large number of introductory texts. The purposes of the present te are, however, somewhat different from those of most introductory topology texts. First, we ha attempted early in the book to lead the reader through a number of nontrivial applications of metr space topology to analysis, so that the relevance of topology to analysis is apparent both with mo immediacy and also on a deeper level than is commonly the case. Second, in the treatment of topi from elementary algebraic topology later in the book, we have concentrated upon results with concre geometric meaning and have presented comparatively little algebraic formalism; at the same tim however, we have provided proofs of some highly nontrivial results (e.g., the noncontractibility of S These goals have been accomplished by treating homotopy theory without considering homolo theory. Thus the reader can immediately see important applications without undertaking t development of a large formal program. We hope that these applications, besides having intrins interest, will lead the reader toward the detailed study of algebraic topology with the feeling th putting its methods on a formal, general basis is well worthwhile. The metric space and point-s topology material occupies the first two chapters, the algebraic topological material the remaining tw chapters.

This book arose from our experiences in teaching introductory courses in various aspects topology to upper division undergraduate students and beginning graduate students. All the materi was found accessible by the students, but it could not all be covered in a single one-term course. T book is arranged so that considerable flexibility in the choice of topics is possible. Chapter II depen on Chapter I only for motivation, and Chapters III and IV depend on only the most basic material Chapter II. In fact, by restricting attention to homotopy groups of metric spaces, Chapters III and can be studied after only the basic material (e.g., compactness, continuity) of Chapter I. Thus a sho course with strong analytical emphasis can easily be constructed by using Chapters I and II in the entirety and only a few topics from Chapters III and IV; a short course emphasizing the geometric a algebraic parts of the subject can be constructed by using little of Chapters I and II and then Chapte III and IV in toto. In all cases, starred sections, which are in general of somewhat greater difficul than the other material, can be omitted without detriment to understanding of later portions of the te

In the strict logical sense, the book is almost independent of prior mathematical knowledge. On ~~familiarity with the real numbers and with some basic set theory, such as countability~~ an uncountability, is needed in this strict sense. Readers who are not to some extent familiar with th subject usually called “rigorous calculus” (e.g., ε – δ definitions of limits and continuity) may find th material a strain on their capacity to absorb abstraction without concrete motivation. Readers famili with rigorous calculus will recognize easily that Chapter I and Chapter II are generalized expressio of phenomena that occur in concrete form in rigorous calculus. Throughout, readers are urged construct for themselves many concrete illustrations of the general definitions and results. Topolog is a subject that, in spite of its appearance of abstractness, is deeply rooted in the concrete an geometrically comprehensible world; the development of intuitive insight by consideration of thes roots is a vital part of learning topology, but it is a part that one must do largely for oneself. In th connection also, we strongly urge the reader to do as many of the exercises as possible. Topology is much a mode of thought as it is a body of information, and mastery of the mode of thought cannot b really well developed by passive reading alone. This remark applies to most mathematics, but it seem to us to apply especially strongly to the learning of topology. Thus the exercises are to us an integr part of the text, and readers should so treat them. In any case, it is more fun to play the game than ju to learn the rules.

Cross references to lemmas, theorems and exercises are handled with notations such as “III.4.7” “5.7.” Thus “Theorem III.4.7” refers to Theorem 7 of Chapter III, Section 4, and “Exercise 5.7” refe to Exercise 7 of Section 5 of the Chapter in which the reference occurs. A list of special symbols given at the end of the book, as are various bibliographical references.

We thank with pleasure the many students and colleagues at UCLA who gave us encourageme and valuable comments on the preliminary lecture notes from which this book evolved over the pa eight years. We especially thank Laurie Beerman for her fine job typing the various revisions of th manuscript.

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Metric Spaces

ON

The ideas of “metric” and “metric space,” which are the subject matter of this chapter, are abstractio of the concept of distance in Euclidean space. These abstractions have turned out to be particular fundamental and useful in modern mathematics; in fact, the aspects of the Euclidean idea of distan retained in the abstract version are precisely those that are most useful in a wide range mathematical activities. The determination of this usefulness was historically a matter of experien and experiment. By now, the reader can be assured, the mathematical utility of the metric-spa information developed in this chapter entirely justifies its careful study.

Sections 1 through 6 of this chapter are devoted to the basic definitions and main theorems abo metric spaces in general. Among the theorems established, two are especially substantial: the resu called the Baire Category Theorem, in Section 2, and the equivalence of compactness and sequenti compactness, in Section 5. The material in these first six sections is basic to modem analysis.

Sections 7 through 9 treat more specialized topics. Section 7 introduces some special classes metric spaces—the normed linear spaces and Banach spaces—that are particularly important applications. These spaces have not only the abstract idea of distance common to all metric spaces b also a vector-space structure that interacts with the distance idea in a desirable fashion. In Section an important result about functions from metric spaces to themselves, the Contraction Mappi Theorem, is proved. This result is applied to obtain solutions to various specific problems of analysi the solution of certain integral equations and the proof of the existence of solutions to certa differential equations (the Cauchy-Picard Theorem). In Section 9, the idea of differentiability f functions on normed linear spaces is introduced. This derivative concept, known as the Frech derivative, is used to develop analogues for these general spaces of the standard inverse and implic function theorems of vector calculus, which are themselves incidentally proved in the process.

Sections 7 through 9 are not essential to an understanding of the remainder of the text, and th can thus be omitted or deferred. Nevertheless, we urge the reader at least to glance at the material these sections to gain some insight into the scope and power of the general metric-space methods.

Most of the ideas about metric spaces in general are motivated by geometric ideas about sets in or $\mathbb{R}^n$, $n > 1$. Since this is true in general, explicit statement to this effect is usually omitted in t specific discussions that follow. The reader should nonetheless consider, each time a new idea occur what geometric meaning the idea has for sets in Euclidean space. The concrete pictures thus forme are very helpful in developing intuition about the general metric-space situation, even though son caution is, as always in abstract mathematics, necessary to ensure that intuition does not lead o astray.

1. OPEN AND CLOSED SETS

A *metric* on a set X is a real-valued function d on $X \times X$ that has the following properties:

$$(1.1) \qquad d(x,y) \geq 0, \qquad x,y \in X,$$

$$(1.2) \qquad d(x,y) = 0 \quad \text{if and only if} \quad x = y,$$

$$(1.3) \qquad d(x,y) = d(y,x), \qquad x,y \in X,$$

$$(1.4) \qquad d(x,z) \leq d(x,y) + d(y,z), \qquad x,y,z \in X.$$

The idea of a metric on a set X is an abstract formulation of the notion of distance in Euclidea space. The intuitive interpretation of property (1.4) is particularly suggestive. This property is th abstract formulation of the fact that the sum of the lengths of two sides of a triangle is greater than equal to the length of the third side. Consequently (1.4) is referred to as the *triangle inequality*.

A *metric space* (X,d) is a set X equipped with a metric d on X. Sometimes we suppress mention the metric d and refer to X itself as being a metric space.

If (X,d) is a metric space and Y is a subset of X, then the restriction d' of d to $Y \times Y$ is clearly metric on Y. The metric space (Y,d') is called a *subspace* of (X,d).

The set of real numbers $\mathbb{R}$, with the usual distance function

$$d(x,y) = |x - y|, \quad x, y \in \mathbb{R},$$

is a metric space since properties (1.1) through (1.4) all hold. More generally, the n-dimension Euclidean space $\mathbb{R}^n$, consisting of all n-tuples $x = (x_1, \ldots, x_n)$ of real numbers, becomes a metr space when endowed with the metric

$$(1.5) \qquad d(x,y) = \left[\sum_{j=1}^{n} (x_j - y_j)^2\right]^{1/2}, \qquad x,y \in \mathbb{R}^n.$$

Actually, it is not immediately clear that (1.5) defines a metric. The verification of properties (1.1 (1.2), and (1.3) is straightforward, but the verification of (1.4) requires some effort, and a proof outlined in Exercise 3. We shall be especially interested in the metric spaces $\mathbb{R}$ and $\mathbb{R}^n$ and in the subspaces.

Note that the cases $n = 2$ and $n = 3$ correspond to the spaces of Euclidean plane geometry and sol geometry, respectively. In this setting, our definition of distance is adopted from the statement of th Theorem of Pythagoras. In mathematics, many good theorems are eventually converted in definitions.

As another example of a metric space, let S be any set and let $B(S)$ denote the set of bounded rea valued functions on S. Endowed with the metric

$$(1.6) \qquad d(f,g) = \sup\{|f(s) - g(s)| : s \in S\},$$

$B(S)$ becomes a metric space. The verification of (1.1) through (1.4) in this case is left to the read (Exercise 5).

Any set X can be made into a "discrete" metric space by associating with X the metric d defined b

$$(1.7) \qquad d(x,y) = \begin{cases} 1, & x \neq y, \\ 0, & x = y. \end{cases}$$

The verification that d is indeed a metric is also left to the reader (Exercise 2).

Let (X,d) be any metric space. The *open ball* $B(x;r)$ with center $x \in X$ and radius $r > 0$ is defin by

$$B(x;r) = \{y \in X : d(x,y) < r\}.$$

The balls centered at x form a nested family of subsets of X that increase with r, that is, $B(x\ ;r_1)$ $B(x;r_2)$ if $r_1 \le r_2$. Furthermore,

$$\bigcup_{r>0} B(x;r) = X,$$

and, because of (1.2),

$$\bigcap_{r>0} B(x;r) = \{x\}.$$

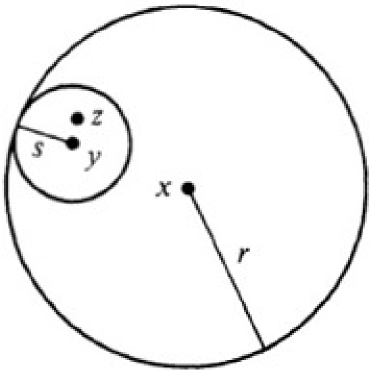
Let Y be a subset of X. A point $x \in X$ is an *interior point* of Y if there exists $r > 0$ such that $B(x;r)$ Y. The set of interior points of y is the *interior* of Y, and it is denoted by int(Y). Note that every interi point of Y belongs to Y:

$$\text{int}(Y) \subseteq Y.$$

A subset Y of X is *open* if every point of Y is an interior point of Y, that is, if int(Y) = Y. In particula the empty set Ø and the entire space X are open subsets of X.

1.1 Theorem: Any open ball $B(x,r)$ in a metric space X is an open subset of X.

Proof: Let $y \in B(x;r)$. It suffices to find some open ball centered at y that is contained in $B(x,r)$. Let $= r - d(x,y)$. Then $s > 0$. If $z \in B(y;s)$, i.e., $d(y,z) < s$, then $d(x,z) \le d(x,y) + d(y,z) < d(x,y) + s = r$, so th $z \in B(x;r)$. Consequently $B(y;s) \subseteq B(x;r)$. □



1.2 Theorem: Let Y be a subset of a metric space X. Then int(Y) is an open subset of X. In oth words, int(int(Y)) = int(Y).

Proof: Let $x \in$ int(Y). Then there exists $r > 0$ such that $B(x;r) \subset Y$. It suffices to show that $B(x;r)$ int(Y).

Since $B(x;r)$ is open, there is for each $y \in B(x;r)$ an open ball $B(y;s)$ contained in $B(x;r)$. particular, each $B(y;s)$ is contained in Y, so that each $y \in B(x;r)$ belongs to int(Y). □

1.3 Theorem: The union of a family of open subsets of a metric space X is an open subset of X.

Proof: Let $\{U_\alpha\}_{\alpha \in A}$ be a family of open subsets of X and let $U = \cup_{\alpha \in A} U_\alpha$. Suppose $x \in U$. Th there is some index α such that $x \in U_\alpha$. Since U_α is open, there exists some $r > 0$ such that $B(x;r)$ U_α. Then $B(x;r) \subset U$, so that x is an interior point of U. Since this is true for all $x \in U$, U is open. □

1.4 Theorem: A subset U of a metric space X is open if and only if U is a union of open balls in X.

Proof: By Theorems 1.1 and 1.3, any set that is a union of open balls is open. On the other han suppose that U is an open subset of X. For each $x \in U$, there then exists $r(x) > 0$ such that $B(x;r(x))$ U. Utilizing U as an index set for a union, we obtain an expression

$$U = \bigcup_{x \in U} B(x;r(x))$$

for U as a union of open balls. □

In general, the union in Theorem 1.4 will have to be infinite. For example, the set $\{(x_1, x_2) \in \mathbb{R}^2$ $0 < x_1 < 1, 0 < x_2 < 1\}$ is open but is not a finite union of open balls.

1.5 Theorem: The intersection of any finite number of open subsets of a metric space is open.

Proof: Let $U_1, \ldots, U_m$ be open subsets of X and let $U = U_1 \cap \ldots \cap U_m$. Let $y \in U$. Since each U_j open, there exist $r_j > 0$ such that

$$B(y;r_j) \subset U_j, \quad 1 \le j \le m.$$

Set $r = \min(r_1, \ldots, r_m)$. Then $r > 0$ (the minimum of a *finite* set of positive numbers is a positi number). Since $B(y;r) \subset U_j$, $1 \le j \le m$, we obtain $B(y;r) \subset U$. It follows that U is open. □

The finiteness assumption in the theorem just given is essential. In $\mathbb{R}^1$, the intersection of ope balls $\bigcap_{i=1}^{\infty} B(0;1/i)$ is the set $\{0\}$ consisting of 0 only. This set is not open in $\mathbb{R}^1$.

A subset of X that is open in a subspace of X need not be open in X. For instance, an open interv (a,b) on the real line $\mathbb{R}$ is an open subset of $\mathbb{R}$. However, $\mathbb{R}$ can be regarded as a subspace of the plan $\mathbb{R}^2$ (by identifying $\mathbb{R}$ with the x-axis in $\mathbb{R}^2$), and then the interval is not an open subset of $\mathbb{R}^2$. What true is the following.

1.6 Theorem: Let Y be a subspace of a metric space X. Then a subset U of Y is open in Y if and on if $U = V \cap Y$ for some open subset V of X.

Proof: Because the metric $d' : Y \times Y \to R$ is the restriction to $Y \times Y$ of the metric $d : X \times X \to R$ on the open ball in the metric space Y with center $y \in Y$ and radius $r > 0$ is just the intersection $B(y;r) \cap$ of Y and the open ball $B(y;r)$ in X. If V is open in X, then for each $y \in V \cap Y$, there exists $r > 0$ such th $B(y;r) \subset V$. Then the open ball in Y centered at y with radius r is contained in $V \cap Y$ = Consequently each $y \in V \cap Y$ is an interior point of $V \cap Y$ in the subspace Y, so that $U = V \cap Y$ is op in Y.

Conversely, suppose that U is an open subset of the subspace Y. Let $y \in U$. Then there exists $r(y)$ 0 such that the open ball $B(y;r(y)) \cap Y$ in Y is contained in U. Then the open subset

$$V = \bigcup_{y \in U} B(y;r(y))$$

of X satisfies $V \cap Y \subset U$. Since each $y \in U$ belongs to V, we obtain $V \cap Y = U$. □

Let Y be a subset of a metric space X. A point $x \in X$ is *adherent* to Y if for all $r > 0$,

$$B(x;r) \cap Y \neq \emptyset.$$

The *closure* of Y, denoted by $\overline{Y}$, consists of all points in X that are adherent to Y. Evidently each poi of Y is adherent to Y, so that

$$Y \subset \overline{Y}.$$

The subset Y is *closed* if $Y = \overline{Y}$. In particular, the empty set $\emptyset$ and the entire space X are clos subsets of X.

Intuitively speaking, a subset is closed if it contains all its boundary points. (For a precise versio of this idea, see Exercise 14.) For example, the union of a circle in $\mathbb{R}^2$ and its inside is a closed set. any points of the circle are deleted from this union, however, then the resulting set is not closed.

1.7 Theorem: If Y is a subset of a metric space X, then the closure of Y is closed, that is, $\overline{\overline{Y}} = \overline{Y}$.

Proof: Since $\overline{Y} \subset \overline{\overline{Y}}$, it suffices to obtain the reverse inclusion, $\overline{\overline{Y}} \subset \overline{Y}$.

Let $x \in \overline{\overline{Y}}$ and let $r > 0$. It suffices to show that $B(x;r) \cap Y \neq \emptyset$. Since x is adherent to $\overline{Y}$, there is point $z \in B(x;r/2) \cap \overline{Y}$. Since z is adherent to Y, there exists a point $y \in B(z;r/2) \cap Y$. Then

$$d(x;y) \leq d(x,z) + d(z,y) < \frac{r}{2} + \frac{r}{2} = r,$$

so that $y \in B(x;r) \cap Y$ and the intersection is not empty. □

1.8 Theorem: A subset Y of a metric space X is closed if and only if the complement of Y is open.

Proof: The set-theoretic difference of two sets U and V consists of those points in U that do n belong to V. Here, and forevermore, the notation used for this difference is $U \backslash V$.

Suppose first that Y is closed. For each $x \in X \backslash Y$, there then exists $r > 0$ such that $B(x;r) \cap Y =$ Then $B(x;r) \subset X \backslash Y$, so that x is an interior point of $X \backslash Y$, and $X \backslash Y$ is open.

Conversely, suppose that $X \backslash Y$ is open. For each $x \in X \backslash Y$, there then exists $r > 0$ such that $B(x;r)$ $X \backslash Y$. Hence $B(x;r) \cap Y = \emptyset$ and $x \notin \overline{Y}$. It follows that $Y = \overline{Y}$. □

Despite Theorem 1.8, it should be noted that being open and being closed are not opposite notion The empty set is both open and closed, and so is the entire space X. In a discrete metric space, eve set is both open and closed. On the other hand, the semiopen interval $(0,1]$, regarded as a subset of is neither open nor closed.

1.9 Theorem: The intersection of any family of closed sets is closed. The union of any finite fami of closed sets is closed.

Proof: The statement on intersections follows from Theorems 1.3 and 1.8, together with the identity

$$X \setminus \bigcap_{\alpha \in A} E_\alpha = \bigcup_{\alpha \in A} (X \setminus E_\alpha),$$

valid for any family $\{E_\alpha\}_{\alpha \in A}$ of subsets of X (Exercise 1). Indeed, if each E_α is closed, then each $X \setminus E_\alpha$ is open, so that the union of the $X \setminus E_\alpha$ is open and the intersection of the E_α is closed. The statement on unions follows from Theorems 1.5 and 1.8, together with the identity (Exercise 1)

$$X \setminus \bigcup_{\alpha \in A} E_\alpha = \bigcap_{\alpha \in A} (X \setminus E_\alpha). \square$$

A sequence $\{x_n\}_{n=1}^{\infty}$ in a metric space X *converges to* $x \in X$ if

$$\lim_{n \to \infty} d(x_n, x) = 0.$$

In this case, x is the *limit* of $\{x_n\}$ and we write $x_n \to x$, or

$$\lim_{n \to \infty} x_n = x.$$

1.10 Lemma: The limit of a convergent sequence in a metric space is unique.

Proof: Suppose that x and $y \in X$ are both limits of a sequence $\{x_n\}$ in X. Then for all n,

$$d(x,y) \leq d(x, x_n) + d(x_n, y).$$

As n tends to infinity, the right-hand side tends to 0, so that $d(x,y) = 0$. Consequently, $x = y$. $\square$

1.11 Theorem Let Y be a subset of the metric space X, Then $x \in X$ is adherent to Y if and only if there is a sequence in Y that converges to x.

Proof: If there is a sequence in Y that converges to x, then every open ball centered at x contains points of the sequence, so that x is adherent to Y. Conversely, suppose that x is adherent to Y. For each integer $n \geq 1$, there exists then some point $x_n \in B(x;1/n) \cap Y$. The sequence $\{x_n\}_{n=1}^{\infty}$ then satisfies $d(x,x_n) < 1/n \to 0$, so that x_n converges to x. $\square$

EXERCISES

1. Let U, V, and W be subsets of some set. Recall that $U \setminus V$ consists of all points in U that do not belong to V.
   (a) Prove that $(U \cup V) \setminus W = (U \setminus W) \cup (V \setminus W)$.
   (b) Prove that $(U \cap V) \setminus W = (U \setminus W) \cap (V \setminus W)$.
   (c) Does $U \setminus (V \setminus W)$ coincide with $(U \setminus V) \setminus W$? Justify your answer by proof or counterexample.
   (d) Prove the two set-theoretic identities used in the proof of Theorem 1.9.
2. Show that (1.7) defines a metric on X. Show that every subset of the resulting metric space

both open and closed.

---

3. (a) Show that if $a, b, c \in \mathbb{R}$ are such that for all $\lambda \in \mathbb{R}$, $a\lambda^2 + b\lambda + c \geq 0$, then $b^2 - 4ac \leq$
   Hint: Find the minimal value of the polynomial in λ.

   (b) Show that for any $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in \mathbb{R}^n$,

$$\sum_{i=1}^{n} x_i y_i \leq \left(\sum_{i=1}^{n} x_i^2\right)^{1/2} \left(\sum_{i=1}^{n} y_i^2\right)^{1/2}.$$

Remark: This is a version of the important *Cauchy-Schwarz inequality*. (Perhaps the most comm spelling error in mathematics is to replace "Schwarz" by "Schwartz.") For the proof, apply pa (a) to

$$\Sigma_{i=1}^{n} (x_i - \lambda y_i)^2.$$

   (c) Using the Cauchy-Schwarz inequality, show that the function d defined by (1.5) satisfies t triangle inequality.

4. Show that the semiopen interval $(0,1]$ is neither open nor closed in $\mathbb{R}$.
5. Show that (1.6) defines a metric on the space $B(S)$ of bounded real-valued functions on a set S

For the following exercises let (X,d) be a metric space.

6. Prove that the interior of a subset Y of X coincides with the union of all open subsets of X th are contained in Y. (Thus the interior of Y is the largest open set contained in Y.)
7. Prove that the closure of a subset Y of X coincides with the intersection of all closed subsets X that contain Y. (Thus the closure of Y is the smallest closed set containing Y.)
8. A set of the form $\{y \in X : d(x,y) \leq r\}$ is called a *closed ball*. Show that a closed ball is a clos set. Is the closed ball $\{ y \in X : d(x,y) \leq r\}$ always the closure of the open ball $B(x;r)$? What if ^{n}? Prove your answers.
9. Let Y be a subspace of X and let S be a subset of Y. Show that the closure of S in Y coincid with $\overline{S} \cap Y$, where $\overline{S}$ is the closure of S in X.
10. A point $x \in X$ is a *limit point* of a subset S of X if every ball $B(x;r)$ contains infinitely ma points of S. Show that x is a limit point of S if and only if there is a sequence $\{x_j\}_{j=1}^{\infty}$ in S su that $x_j \to x$ and $x_j \neq x$ for all j. Show that the set of limit points of S is closed.
11. A point $x \in S$ is an *isolated point* of S if there exists $r > 0$ such that $B(x;r) \cap S = \{x\}$. Show th the closure of a subset S of X is the disjoint union of the limit points of S and the isolated poin of S.
12. Two metrics on X are *equivalent* if they determine the same open subsets. Show that two metri d,ρ on X are *equivalent* if and only if the convergent sequences in (X,d) are the same as t convergent sequences in (X,ρ).
13. Define ρ on $X \times X$ by

$$\rho(x,y) = \min(1,d(x,y)), \quad x, y \in X.$$

    Show that ρ is a metric that is equivalent to d. (Hence every metric is equivalent to a bound metric.)
14. The *boundary* ∂E of a set E is defined to be the set of points adherent to both E and t

complement of E,

$$\partial E = \overline{E} \cap \overline{(X \backslash E)}.$$

Show that E is open if and only if $E \cap \partial E$ is empty. Show that E is closed if and only if $\partial E \subseteq E$

2. COMPLETENESS

The definition of a convergent sequence involves not only the sequence itself but also the limit of th sequence. We wish to develop a notion of convergence that is intrinsic to the sequence, that is, on that does not require having at hand an object that can be called the limit of the sequence. For this, w make the following definition.

A sequence $\{x_n\}_{n=1}^{\infty}$ in a metric space X is a *Cauchy sequence* if

$$\lim_{m,n\to\infty} d(x_n, x_m) = 0,$$

that is, if for each $\varepsilon > 0$, there is an N such that $d(x_n, x_m) < \varepsilon$ for all $n, m \geq N$.

2.1 Lemma: A convergent sequence is a Cauchy sequence.

Proof: Suppose that $\{x_n\}$ converges to x. Then

$$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m)$$

and the right-hand side tends to zero as $n, m \to \infty$. □

2.2 Lemma: If $\{x_n\}$ is a Cauchy sequence and if there is a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}$ of $\{x_n\}$ th converges to x, then $\{x_n\}$ converges to x.

Proof: Let $\varepsilon > 0$. Choose $k > 0$ so large that $d(x_n, x_m) < \varepsilon/2$ whenever $n, m \geq k$. Choose $l > 0$ so lar that $d(x_{n_j}, x) < \varepsilon/2$ whenever $j \geq l$. Set $N = \max(k, n_l)$. If $m \geq N$ and $n_j \geq N$, then

$$d(x_m, x) \leq d(x_m, x_{n_m}) + d(x_{n_j}, x) < \varepsilon/2 + \varepsilon/2 = \varepsilon. \ \square$$

A metric space X is *complete* if every Cauchy sequence in X converges.

Not every metric space is complete. For instance, the set $X = \{x \in \mathbb{R} : 0 < x < 1\}$ with metric $d(x,$ $= |x - y|$ is not complete since $\{1/n\}$ is a Cauchy sequence that converges to no point (of X). Howeve many important metric spaces are complete; in particular, $\mathbb{R}^n$ is complete for all $n = 1, 2, 3, \ldots$. Th completeness will be established in the next section.

The space $\{x \in \mathbb{R} : 0 < x < 1\}$, though not complete, is a subspace of the complete metric spac $\tilde{X} = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$. (That $\tilde{X}$ is complete will follow from the completeness of $\mathbb{R}$ an Theorem 2.3.) This example illustrates the general situation: every metric space X may be regarded a subspace of a complete metric space $\tilde{X}$ in such a way that $\overline{X} = \tilde{X}$ (Exercise 7). Thus an noncomplete metric space can be thought of as a complete metric space with certain points deleted.

2.3 Theorem: A closed subspace of a complete metric space is complete.

Proof: Let Y be a closed subspace of a complete space X and let $\{y_n\}$ be a Cauchy sequence in Then $\{y_n\}$ is also a Cauchy sequence in X. Since X is complete, there exists $x \in X$ such that converges to x in X. Since Y is closed, x must belong to Y. Consequently $\{y_n\}$ converges to x in Y. □

2.4 Theorem: A complete subspace Y of a metric space X is closed in X.

Proof: Suppose $x \in X$ is adherent to Y. By Theorem 1.11, there is a sequence $\{y_n\}$ in Y that converg to x. By Lemma 2.1, $\{y_n\}$ is a Cauchy sequence in X; hence it is also a Cauchy sequence in Y. Since is complete, $\{y_n\}$ converges to some point $y \in Y$. Since limits of sequences are unique, $y = x$ and belongs to Y. Consequently Y is closed. □

Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions from a set S to a metric space X and let f be a function fro S to X. The sequence $\{f_n\}$ *converges uniformly* to f on S if for each $\varepsilon > 0$ there exists an integer N su that $d(f_n(s), f(s)) < \varepsilon$ for all integers $n \geq N$ and for all $s \in S$. A sequence $\{f_n\}$ of functions from S to X a *Cauchy sequence of functions* if for each $\varepsilon > 0$ there exists an integer N such that

$$d(f_n(s), f_m(s)) < \varepsilon \quad \text{all } s \in S,\ n, m \geq N.$$

The idea of a Cauchy sequence of functions can be related to the idea of a Cauchy sequence in a metr space by defining an appropriate metric on the set of all functions from S to X (Exercise 6).

2.5 Theorem: Let S be a set, and let X be a complete metric space. If $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequen of functions from S to X, then there exists a function f from S to X such that $\{f_n\}$ converges uniform to f.

Proof: For each fixed $s \in S$, $\{f_n(s)\}_{n=1}^{\infty}$ is a Cauchy sequence in X. Since X is complete, $\{f_n(s$ converges to some point of X, which we define to be $f(s)$. Thus f is a function from S to X.

Let $\varepsilon > 0$. Choose an integer $N \geq 1$ such that $d(f_n(s), f_m(s)) < \varepsilon$ for all $s \in S$ and all $n, m \geq N$. Then

$$\begin{aligned} d(f_n(s), f(s)) &\leq d(f_n(s), f_m(s)) + d(f_m(s), f(s)) \\ &< \varepsilon + d(f_m(s), f(s)) \end{aligned}$$

whenever $n, m \geq N$. Letting m tend to ∞, we obtain

$$d(f_n(s), f(s)) \leq \varepsilon, \quad n \geq N,$$

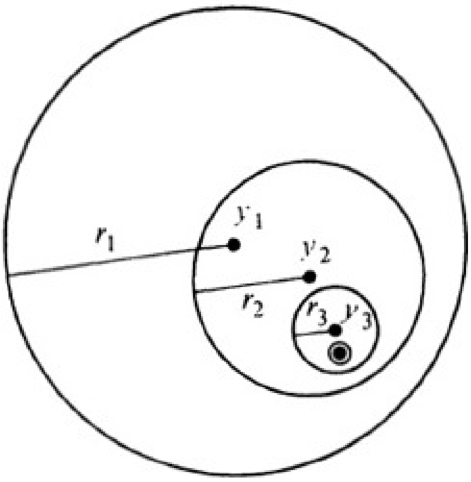
for all. $s \in S$. Consequently $\{f_n\}$ converges uniformly to f on S. □

A subset T of a metric space X is *dense* in X if $\overline{T} = X$. The following theorem involving th concept has many important applications.

2.6 Theorem *(Baire Category Theorem):* Let $\{U_n\}_{n=1}^{\infty}$ be a sequence of dense open subsets of complete metric space X. Then $\bigcap_{n=1}^{\infty} U_n$ is also dense in X.

Proof: Let $x \in X$ and let $\varepsilon > 0$. It suffices to find $y \in B(x;\varepsilon)$ that belongs to $\bigcap_{n=1}^{\infty} U_n$. Indeed, th every open ball in X meets $\bigcap U_n$, so that $\bigcap U_n$ is dense in X.

Since U_1 is dense in X, there exists $y_1 \in U_1$ such that $d(x,y_1) < \varepsilon$. Since U_1 is open, there exists r_1 0 such that $B(y_1;r_1) \subset U_1$. By shrinking r_1, we can arrange that $r_1 < 1$, and $\overline{B(y_1;r_1)} \subset U_1 \cap B(x,\varepsilon$ The same argument, with $B(y_1;r_1)$ replacing $B(x;\varepsilon)$, produces $y_2 \in X$ and $0 < r_2 < 1/2$ such tha $\overline{B(y_2;r_2)} \subset U_2 \cap B(y_1;r_1)$.



Continuing in this manner, we obtain a sequence $\{y_n\}_{n=1}^{\infty}$ in X and a sequence $\{r_n\}_{n=1}^{\infty}$ or radii suc that $0 < r_n < 1/n$ and

$$\overline{B(y_n;r_n)} \subseteq U_n \cap B(y_{n-1};r_{n-1}). \tag{2.1}$$

It follows that

$$\overline{B(y_n;r_n)} \subset B(y_{n-1};r_{n-1}) \subset \ldots \subset B(y_1;r_1) \subset B(x;\varepsilon). \tag{2.2}$$

The nesting property (2.2) shows that $y_m \in B(y_n;r_n)$ if $m > n$, so that $d(y_m, y_n) < r_n \to 0$ as $m, n \to$ Consequently $\{y_m\}$ is a Cauchy sequence. Since X is complete, there exists $y \in X$ such that $y_m \to$ Since $y_m \in B(y_n;r_n)$ for $m > n$, we obtain $y \in \overline{B(y_n;r_n)}$. By (2.2) $y \in B(x;\varepsilon)$, and by (2.1) $y \in U_n$; th for all n, so that $y \in \bigcap_{n=1}^{\infty} U_n$. □

The hypothesis of completeness in the Baire Category Theorem is crucial. For example, consid the subspace of $\mathbb{R}$ consisting of the rational numbers $\mathbb{R}_0$. Arrange the rational numbers in a sequen $\{s_n\}_{n=1}^{\infty}$ and set $U_n = \mathbb{R}_0 \backslash \{s_n\}$. Then each U_n is a dense open subset of $\mathbb{R}_0$. However, $\bigcap_{n=1}^{\infty} U_n$ empty.

A subset Y of X is *nowhere dense* if $\overline{Y}$ has no interior points, that is, if

$$\operatorname{int}(\overline{Y}) = \varnothing.$$

Evidently Y is nowhere dense if and only if $X \backslash \overline{Y}$ is a dense open subset of X. By taking complemen of the sets in the Baire Category Theorem, we obtain the following equivalent version of the theorem

2.7 Corollary: Let $\{E_n\}_{n=1}^{\infty}$ be a sequence of nowhere dense subsets of a complete metric space Then $\bigcup_{n=1}^{\infty} E_n$ has empty interior.

Proof: Apply the Baire Category Theorem to the dense open sets $U_n = X \setminus \overline{E}_n$. □

An explanation of the nomenclature is perhaps in order. A subset of a metric space X is of the *fir* *category* (or *meager*) if it is the countable union of nowhere-dense subsets. A subset that is not of th first category is said to be of the *second category*. The Baire Category Theorem is then equivalent the following statement. In a complete metric space X, the complement of a set of the first category dense in X. The general idea of the terminology is that a nowhere dense set is a really small set an that a countable union of nowhere dense sets is still small in the sense that it has empty interior (b the Baire Category Theorem). Thus a set of the first category is a small set; a set of the secon category is a set that is not small in this sense.

The Baire Category Theorem is a far-reaching generalization of the uncountability of the re numbers. Single-point subsets of the real numbers are nowhere dense, and the uncountability of th real numbers is just the fact that countable unions of such single-point sets cannot be the whole of th real numbers. Thus, the Baire Category Theorem has as a corollary the uncountability of the re numbers, once it is known that the real numbers are complete in their usual metric-space structur This completeness is discussed in the next section.

The applications of the Baire Category Theorem go far beyond proving uncountability, howeve Some additional applications to an entirely different situation are presented in Section 7. Some furth applications to the real numbers are given in Exercises 3.5 to 3.7.

EXERCISES

1. A sequence $\{x_k\}_{k=1}^{\infty}$ in a metric space (X,d) is a *fast Cauchy sequence* if
$$\sum_{k=1}^{\infty} d(x_k,x_{k+1}) < \infty.$$
Show that a fast Cauchy sequence is a Cauchy sequence.
2. Prove that every Cauchy sequence has a subsequence that is a fast Cauchy sequence.
3. Prove that the set of isolated points of a countable complete metric space X forms a den subset of X.
4. Suppose that F is a subset of the first category in a metric space X and E is a subset of F. Pro that E is of the first category in X. Show by an example that E may not be of the first category the metric space F.
5. Prove that any countable union of sets of the first category in X is again of the first category X.
6. Let S be a nonempty set, let (X,d) be a metric space, and let $\mathcal{F}$ be the set of functions from S X. For $f, g \in \mathcal{F}$, define
$$\rho(f,g) = \sup_{s \in S} \min(1, d(f(s),g(s))).$$
Show that ρ is a metric in $\mathcal{F}$. Show that a sequence $\{f_n\}$ converges to f in the metric space $(\mathcal{F},$ if and only if $\{f_n\}$ converges uniformly to f on X. Show that $(\mathcal{F},\rho)$ is complete if and only (X,d) is complete.
7. Let (X,d) be a metric space and let S be the set of Cauchy sequences in S. Define a relation "~ in X by declaring "$\{s_k\} \sim \{t_k\}$" to mean that $d(s_k, t_k) \to 0$ as $k \to \infty$.

(a) Show that the relation "~" is an equivalence relation.

(b) Let $\tilde{X}$ denote the set of equivalence classes of S and let $\tilde{s}$ denote the equivalence class of $s = \{s_k\}_{k=1}^{\infty}$. Show that the function

$$\rho(\tilde{s},\tilde{t}) = \lim_{k\to\infty} d(s_k,t_k), \qquad \tilde{s},\tilde{t} \in \tilde{X}$$

defines a metric on $\tilde{X}$.

define a metric on $\tilde{X}$.

(c) Show that $(\tilde{X},\rho)$ is complete.

(d) For $x \in X$, define $\tilde{x}$ to be the equivalence class of the constant sequence $\{x, x, \ldots\}$. Show that the function $x \mapsto \tilde{x}$ is an isometry of X onto a dense subset of $\tilde{X}$. (By an isometry, we mean that $d(x,y) = \rho(\tilde{x},\tilde{y})$, $x, y \in X$.)

Note: If a complete metric space Y contains X as a dense subspace, we say that Y is a *completion* of X. The space $\tilde{X}$ of Exercise 7 can be regarded as a completion of X by identifying each $x \in X$ with the constant sequence $\{x, x, \ldots\}$. The next part of the exercise shows that the completion of X is unique, up to isometry.

(e) Show that when Y is a completion of X, then the inclusion map $X \to Y$ extends to an isometry of $\tilde{X}$ onto Y.

8. The *diameter* of a nonempty subset E of a metric space (X,d) is defined to be

$$\text{diam}(E) = \sup\{d(x,y) : x, y \in E\}.$$

Show that if $\{E_k\}_{k=1}^{\infty}$ is a decreasing sequence of closed nonempty subsets of a complete metric space whose diameters tend to zero, then $\bigcap_{k=1}^{\infty} E_k$ consists of precisely one point. How much of the conclusion remains true if X is not complete? Can this property be used to characterize complete metric spaces? Justify your answer.

3. THE REAL LINE

The set of real numbers will always be denoted by $\mathbb{R}$. Unless stated otherwise, $\mathbb{R}$ will be regarded as a metric space, with the metric $d(s,t) = |s - t|$, $s, t \in \mathbb{R}$. The set of rational numbers is denoted by $\mathbb{R}_0$.

It is possible to construct the real number system $\mathbb{R}$ from the rational numbers; the construction is a special case of forming the completion of a metric space. The procedure for completing a metric space was outlined in Exercise 2.7. Familiarity with this construction will not be assumed. We shall assume familiarity only with certain properties of the real numbers, which can be taken as axioms.

A first batch of properties involves the algebraic structure of $\mathbb{R}$. These properties assert that the set $\mathbb{R}$, equipped with the operations of addition and multiplication, forms an algebraic entity called a *field*. These properties are just a guarantee that $\mathbb{R}$ follows the usual rules of arithmetic. These arithmetic properties will not be listed here.

The second batch of properties involves the ordering in $\mathbb{R}$. There is a relation "<" on $\mathbb{R}$ that obeys the following laws:

Trichotomy: If $r, s \in \mathbb{R}$, then exactly one of the following three possibilities must hold: $r < s$, $r = s$, or $s < r$.

Transitivity: If $r < s$ and $s < t$, then $r < t$.

Compatibility With Addition: If $r < s$ and $t \in \mathbb{R}$, then $r + t < s + t$.
Compatibility With Multiplication: If $r < s$ and $t > 0$, then $rt < st$.

A field with an ordering satisfying these axioms is a *totally ordered field.*

The properties we have mentioned so far do not characterize $\mathbb{R}$. Indeed, $\mathbb{R}_0$ is also a total ordered field. That is, $\mathbb{R}_0$ satisfies the rules of arithmetic and has a relation "<" (the usual order of $_0$) that satisfies the rules listed. The final property, which characterizes $\mathbb{R}$ up to "isomorphism," is t Least Upper Bound Axiom. (Characterization up to isomorphism means that any other system havi all these properties can be put into one-to-one correspondence with $\mathbb{R}$ in such a way that t correspondence preserves arithmetic operations and order. The fact that the properties give characterize $\mathbb{R}$ in this sense will not be further discussed here.) A subset S of $\mathbb{R}$ is *bounded above* there exists $M \in \mathbb{R}$ such that $s < M$ for all $s \in S$; such an M is an *upper bound* for S. An upper bound for S is a *least upper bound* for S if $M \leq M'$ for any other upper bound M' for S.

Least Upper Bound Axiom: If S is a nonempty subset of $\mathbb{R}$ that is bounded above, then S has a lea upper bound.

Using the Least Upper Bound Axiom, we obtain the following property of $\mathbb{R}$, which is crucial f our purposes.

3.1 Theorem: The real line $\mathbb{R}$, with the usual metric $d(s,t) = |s - t|$, is a complete metric space.

Proof: Let $\{s_k\}_{k=1}^{\infty}$ be a Cauchy sequence in $\mathbb{R}$. Let S be the set of $y \in \mathbb{R}$ such that $s_n < y$ for on finitely many integers n. If $y \in S$ and $z < y$, then evidently $z \in S$. Consequently S includes the enti interval $(-\infty, y]$, just as soon as $y \in S$.

Let $\varepsilon > 0$. Since $\{s_n\}$ is Cauchy, there is an integer N such that $|s_n - s_m| < \varepsilon$ whenever $m, n \geq N$, that all but finitely many terms of the sequence lie in the interval $(s_N - \varepsilon, s_N + \varepsilon)$. In particular, $s_N -$ $\in S$, so that S is not empty. Also, no $t \geq s_N + \varepsilon$ belongs to S, so that $s_N + \varepsilon$ is an upper bound for S. the Least Upper Bound Axiom, S has a least upper bound, call it b. Since $s_N + \varepsilon$ is an upper bound, b $s_N + \varepsilon$. Since $s_N - \varepsilon \in S$, $s_N - \varepsilon \leq b$. Hence $|s_N - b| \leq \varepsilon$. If $m \geq N$, then

$$|s_m - b| \leq |s_m - s_N| + |s_N - b| \leq \varepsilon + \varepsilon = 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, $s_m \to b$ as $m \to \infty$. □

EXERCISES

1. Use the Least Upper Bound Axiom, together with the usual manipulations of algebra identities and inequalities, to prove the following:
   (a) The set $\mathbb{Z}$ of integers is not bounded above.
   (b) For each $\varepsilon > 0$, there exists a rational number $r \in (0,\varepsilon)$.
   (c) If $a, b \in \mathbb{R}$ satisfy $a < b$, then there exists a rational number $s \in (a,b)$.
   (d) The set $\mathbb{R}_0$ of rational numbers is dense in $\mathbb{R}$.
2. Prove that the set of irrational numbers is dense in $\mathbb{R}$.

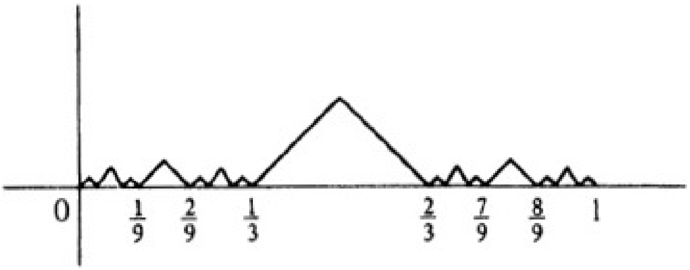
3. Regard the rational numbers $\mathbb{R}_0$ as a subspace of $\mathbb{R}$. Does the metric space $\mathbb{R}_0$ have ar isolated points? Why does this not contradict Exercise 2.3?

4. Prove that every open subset of $\mathbb{R}$ is a union of disjoint open intervals (finite, semi-infinite, infinite).

5. Prove that the set of irrational numbers cannot be expressed as the union of a sequence closed subsets of $\mathbb{R}$.

6. Prove that the set of rational numbers cannot be expressed as the intersection of a sequence open subsets of $\mathbb{R}$.

7. Let E_0 be the closed unit interval [0,1]. Let E_1 be the closed subset of E_0 obtained by removi the open, middle third of E_0, so that $E_1 = [0,1/3] \cup [2/3,1]$. Let E_2 be the closed subset of obtained by removing the open middle thirds of each of the two intervals in E_1, so that consists of four closed intervals, each of length $1/3^2$. Proceeding in this manner, we construct to consist of 2^n closed intervals, and we obtain E_{n+1} by removing the open middle third fro each interval of E_n. The *Cantor set* is defined to be the intersection of the E_n's. Prove tl following:
   (a) The Cantor set is a closed subset of $\mathbb{R}$ with empty interior.
   (b) The Cantor set has no isolated points.
   (c) The Cantor set is uncountable. *Hint:* Use Exercise 2.3.



8. Define explicitly a continuous function f on the unit interval [0,1] which is zero precisely on tl Cantor set and which has a graph as suggested by the figure. Prove that f satisfies the followi condition:

$$|f(s) - f(t)| \leq |s - t|, 0 \geq s, t \geq 1.$$

Remark: This type of condition is called a *Lipschitz condition* on f. Lipschitz conditions will ari again in Section 8.

9. Determine the interior, the closure, the limit points, and the isolated points of each of tl following subsets of $\mathbb{R}$:
   (a) the interval [0,1),
   (b) the set of rational numbers,
   (c) $\{m + n\pi : m$ and n positive integers$\}$,
   (d) $\left\{\frac{1}{m} + \frac{1}{n} : m \text{ and } n \text{ positive integers}\right\}$.

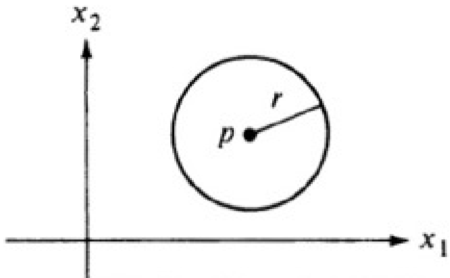
10. Let f be a real-valued function on $\mathbb{R}$. Show that there exist $M > 0$ and a nonempty open subset of $\mathbb{R}$ such that for any $s \in U$, there is a sequence $\{s_n\}$ satisfying $s_n \to s$ and $|f(s_n)| \leq M$, $n \geq 1$.

4. PRODUCTS OF METRIC SPACES

Let $(X_1, d_1), \ldots, (X_n, d_n)$ be metric spaces. The product set $X = X_1 \times \cdots \times X_n$ consists of all n-tupl $(x_1, \ldots, x_n)$, where $x_k \in X_k$, $1 \le k \le n$. There are various ways of making X into a metric space. O possible choice of a metric for X is given by

$$(4.1) \qquad d(x,y) = [d_1(x_1,y_1)^2 + \cdots + d_n(x_n,y_n)^2]^{1/2},$$

where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. This metric on the product stems from the formula for t metric on $\mathbb{R}^n = \mathbb{R} \times \ldots \times \mathbb{R}$ in terms of the metric on the component spaces $\mathbb{R}$. An open ball in $\mathbb{R}^2$ $\mathbb{R} \times \ldots \times \mathbb{R}$ with the metric (4.1) is simply a disc.



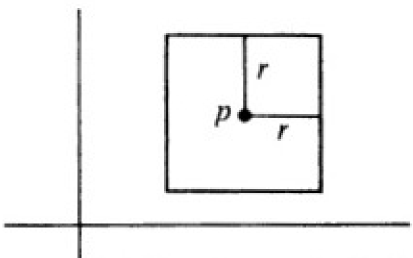
A ball, with metric (4.1)

There are other possible choices of a metric for X, which are convenient for certain problems. Tw of them are given by

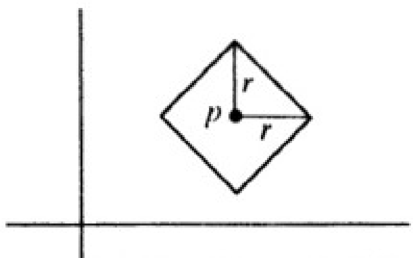
$$(4.2) \qquad d(x,y) = \max(d_1(x_1,y_1), \ldots, d_n(x_n,y_n)),$$

$$(4.3) \qquad d(x,y) = d(x_1,y_1) + \cdots + d_n(x_n,y_n).$$

Open balls in $\mathbb{R} \times \mathbb{R}$ with the metrics (4.2) and (4.3) are squares and diamonds, respectively. dealing with many problems, it is not important which metric is introduced in the product space, long as the metric behaves reasonably. One property that each of the above metrics enjoys is t following.



A ball, with metric (4.2)



A ball, with metric (4.3)

(4.4) A sequence $\{x^{(j)} = (x_k^{(j)})\}_{j=1}^{\infty}$ converges to $x = (x_1, \ldots, x_n)$ in X if and only if for each k the sequence of component entries $\{x_k^{(j)}\}_{j=1}^{\infty}$ converges to x_k in X_k.

The following result shows that all metrics satisfying (4.4) determine the same family of open sets X, and it describes the structure of these open sets explicitly.

4.1 Theorem: Suppose that d is a metric on $X = X_1 \times \cdots \times X_n$ that satisfies (4.4). Then the open se in (X,d) are the unions of product sets of the form $U_1 \times \cdots \times U_n$, where U_j is an open subset of X_j, $1 \le$