

Andrei D. Polyanin
Alexander V. Manzhirov



**HANDBOOK OF
INTEGRAL
EQUATIONS**

SECOND EDITION

 **Chapman & Hall/CRC**
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**HANDBOOK OF
INTEGRAL
EQUATIONS**

SECOND EDITION

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