

Andrei D. Polyanin  
Alexander V. Manzhirov



**HANDBOOK OF  
INTEGRAL  
EQUATIONS**

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**SECOND EDITION**

 **Chapman & Hall/CRC**  
Taylor & Francis Group

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INTEGRAL  
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# HANDBOOK OF INTEGRAL EQUATIONS

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SECOND EDITION

**Andrei D. Polyinin**  
**Alexander V. Manzhirov**

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Taylor & Francis Group  
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