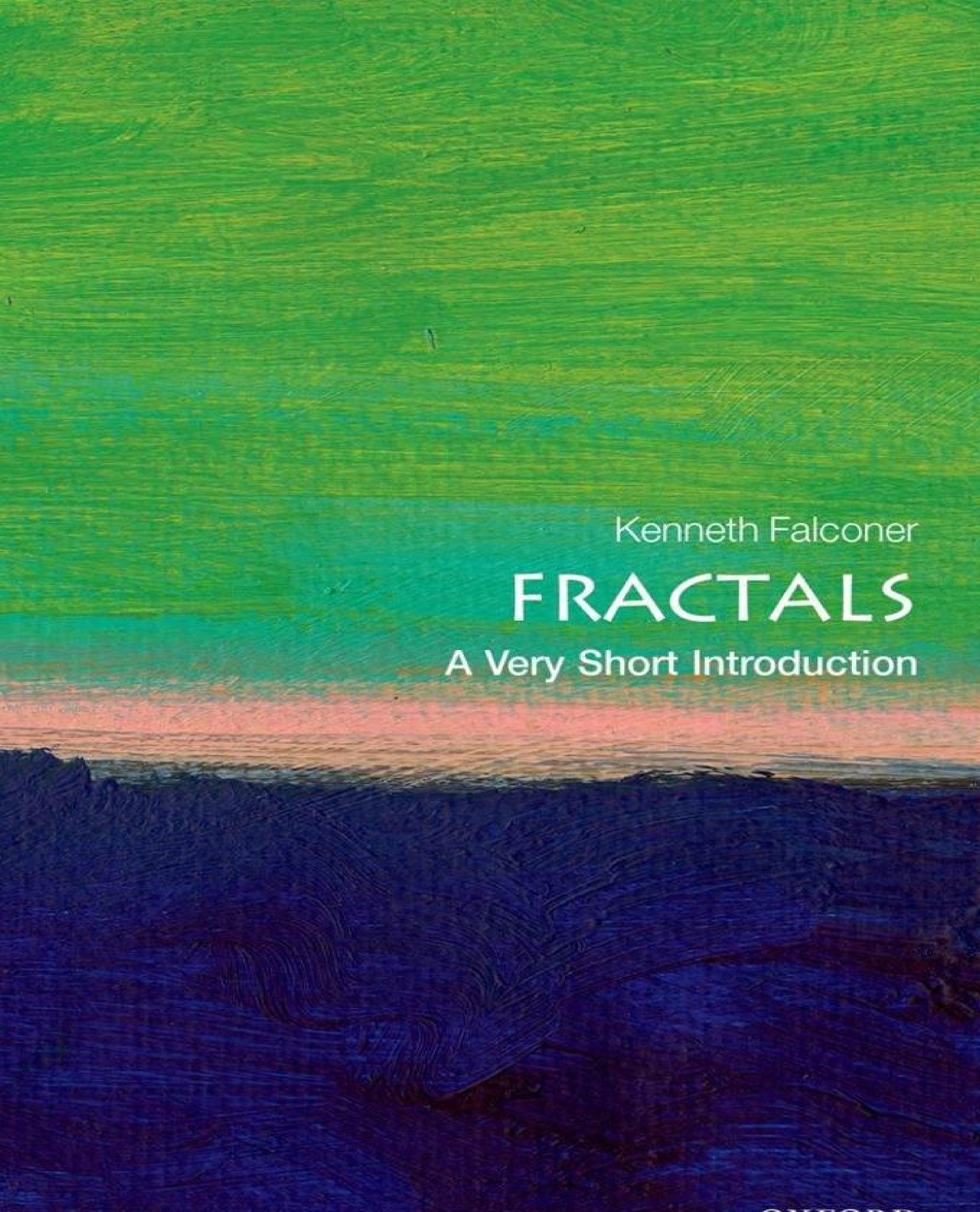

Kenneth Falconer

FRACTALS

A Very Short Introduction

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FRACTALS

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Contents

Preface

To many people, the word ‘geometry’ conjures up circles, cubes, cylinders, and other regular or smooth objects. Familiar artefacts, such as buildings, furniture, or cars, make wide use of such shapes. However, many phenomena in nature and science are anything but regular or smooth. For example, a natural landscape may include bushes, trees, rugged mountains, and clouds, which are far too intricate to be represented by classical geometric shapes.

Surprisingly, apparently complex and irregular objects can often be described in remarkably simple terms. Fractal geometry provides a framework in which a simple process, involving a basic operatio repeated many times, can give rise to a highly irregular result. Fractal constructions can represent natural objects but also give rise to a vast array of other shapes, which may be of extraordinary complexity. The phrase ‘the beauty of fractals’ is often heard, a phrase which reflects the unending intricacy of fractal designs alongside the simplicity which underlies their ever-repeating form. Indee complex but attractive fractal pictures have become an art form in their own right, with exhibitions, competitions, and their use on designer clothing.

The aim of this book is to show, in basic terms, how fractals may be constructed, described, and analysed as geometrical objects and how these ‘mathematical’ fractals relate to the ‘real’ fractals of nature or science. Geometry, particularly fractal geometry, is very much a visual subject, and diagrams and a visual intuition are key to its appreciation. Inevitably some mathematics is required, but this is presented alongside a visual and intuitive interpretation and hopefully will not present too great an obstacle to the dedicated reader prepared to think through the concepts involved. In a couple of places, to keep the mathematics in the main part of the chapters to the minimum, further details ar deferred to an Appendix at the end of the book.

Since fractals became popular in the 1980s, their theory and applications have developed beyond recognition, with fractals pervading many areas of mathematics and science, as well as economics an social science. Fractals are central to a great deal of sophisticated research, which has at its heart the simple ideas that are sketched in this Very Short Introduction. If the reader is able to appreciate some of these ideas then the book will have succeeded in its aim.

I am most grateful to Isobel Falconer, Timothy Gowers, and Emma Ma for reading earlier drafts of this book, to Ben Falconer and Jonathan Fraser for assistance with producing some of the figures, and to Carol Carnegie, Prabhavathy Parthiban, Joy Mellor, and Latha Menon of Oxford University Press for their work in seeing this Very Short Introduction through to publication.

Kenneth Falcon

St Andrews, Scotland, 20

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Chapter 1

The fractal concept

The rise of fractals

Since ancient times, mathematics and science have developed alongside each other, with mathematic used to describe, and often explain, observed natural and physical phenomena. In many areas this marriage has been highly successful, indeed much of what we enjoy in modern life is a consequence of its success. For example, the mathematical methods and laws introduced by Isaac Newton underli the operation of almost everything mechanical, from riding a bicycle to the orbit of a spacecraft. James Clark Maxwell’s equations of electromagnetism hastened the understanding and development of radio communication. A picture on a computer screen can only be created or moved around using mouse because a great deal of geometrical calculation has gone into designing the software.

Nevertheless, there are many phenomena which, although governed by the basic laws of science, wer historically regarded as too irregular or complex to be described or analysed using traditional mathematics. Classical geometry concentrated on smooth or regular objects such as circles, ellipses, cubes, or cones. The calculus, introduced by Newton and Leibniz in the second half of the 17th century, was an ideal tool for analysing smooth objects and rapidly became so central both in mathematics and science that any attempt to consider irregular objects was sidelined. Indeed, many natural phenomena were overlooked, perhaps deliberately, because their irregularity and complexity made them difficult to describe in a form that was mathematically manageable.

It was not until the late 1960s that the study of irregular figures started to develop in a systematic wa largely as a result of efforts by the French-American polymath Benoit Mandelbrot (1924–2010), ofte referred to as ‘the Father of Fractals’. In his 1982 book *The Fractal Geometry of Nature* he wrote: ‘Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line.’ He argued that highly irregular objects should be regarde as commonplace, rather than as exceptional, and moreover that many phenomena from across physic biology, finance, and mathematics have irregularities of a form that are akin to each other. Mandelbr introduced the word *fractal* as a general description for a large class of irregular objects, and highlighted the need for a fractal mathematics to be developed, or in some cases retrieved from isolated forgotten papers.

Since the 1980s fractals have attracted widespread interest. Virtually every area of science has been examined from a fractal viewpoint, with ‘fractal geometry’ becoming a major area of mathematics, both as a subject of interest in its own right and as a tool for a wide range of applications. Fractals have also achieved a popular vogue, with attractive, highly coloured, fractal pictures appearing in magazines, books, and art exhibitions, and even used for scenery in science fiction films. Further public interest has been generated with the widespread use of computers at home and at school, enabling anyone with a basic knowledge of programming to generate intricate fractal pictures by repeatedly applying a simple operation.

Of course, there is always a difference between idealized mathematical objects and the real phenomena that they may represent. A circle has a precise definition as those points on a piece of paper or other flat surface whose distance from a centre exactly equals a given radius, and a sphere consists of those points in space with the same property. We may refer to a coin or wheel as circular, and an orange or the earth as spherical, but these are only approximate descriptions. On close examination, the surface of an orange is dimpled and may be slightly flattened at the top and bottom so it is not quite a sphere, and the earth's surface is covered with hills and valleys. Nevertheless, in th right context it is extremely useful to regard these objects as circles or spheres. If you want to calculate the number of oranges that can be packed into a box it is good enough to assume that they are spherical, and when computing orbits of the earth around the sun or the moon around the earth little is lost by assuming that these bodies are indeed spheres.

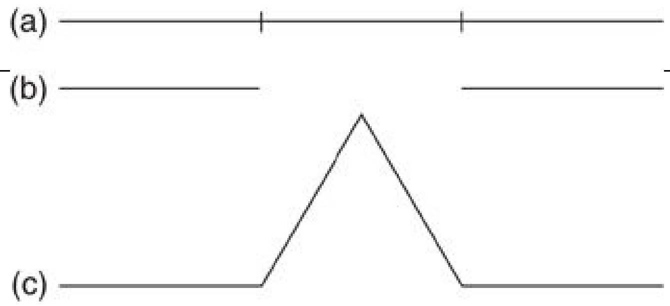
In the same way, we will define fractals in a mathematically exact manner. But we will also encounte natural, physical, and economic phenomena that can usefully be regarded as fractals when viewed ov an appropriate range of scales. These will be thought of as 'real', as opposed to 'mathematical', fractals, with the fractal description inevitably ceasing to be valid if they are examined too closely. This book contains pictures of various fractals, but they are no more than approximations to the exac mathematical objects which possess detail at a far finer scale than anything that can be printed on a page.

A first fractal construction—the von Koch curve

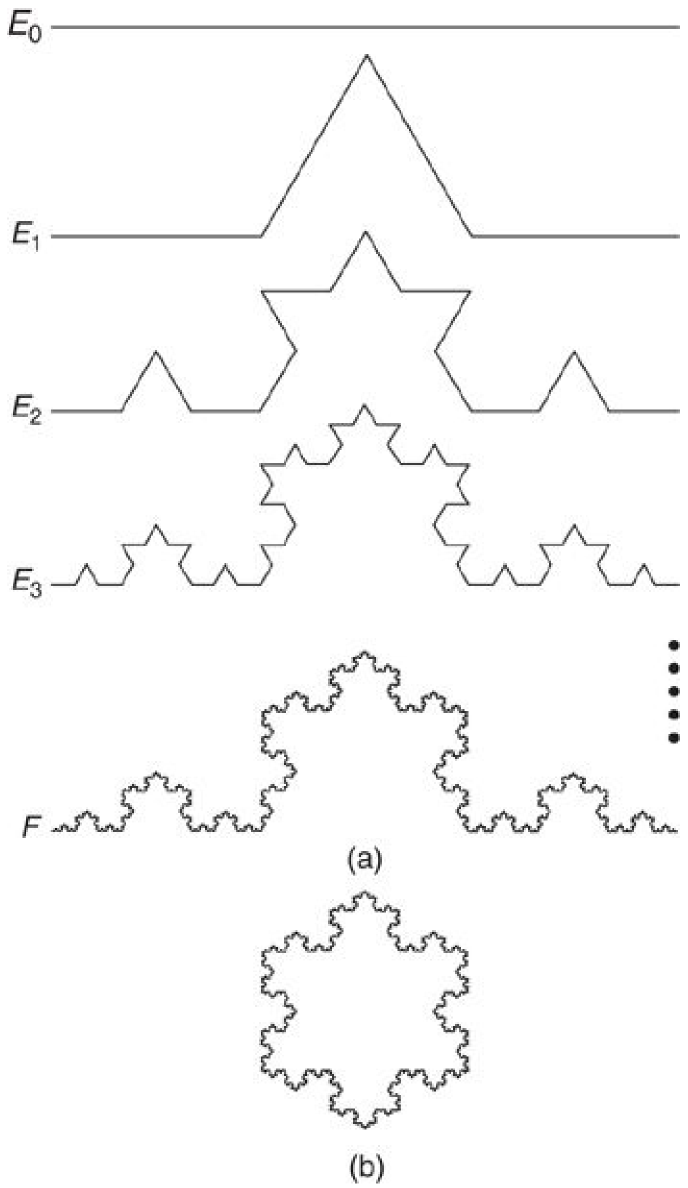
Let's start with a shape that can be drawn, at least very roughly, using just a pencil and eraser.

Take a straight line and divide it into three equal pieces. Erase the middle piece, and replace this by the other two sides of an equilateral (i.e. equal-sided) triangle on the same base. This gives a chain of four shorter, joined-up, straight-line segments: see Figure 1. Now do exactly the same thing with eac of these four pieces: remove the middle thirds and replace by the other two sides of the equilateral triangles on the same base, to get a chain of 16 straight pieces. And now repeat this process again and again. The stages of this construction (which of course have been drawn on a computer rather than by hand) are shown in Figure 2(a). In principle we carry on forever with this procedure, but before long the figure becomes indistinguishable to the eye from the 'curve' labelled *F*. This is known as the *von Koch curve* introduced and studied by the Swedish mathematician Helge von Koch (1870–1924) in 1906. (Note that the word *curve* is used here simply to mean a path that can be traced from one end to the other, without any implication of smoothness.)

Look more closely at the von Koch curve. On zooming in on the curve, however much we magnify it, irregularities in its shape will always be apparent—indeed the curve contains tiny von Koch curves just as wiggly as the original, see Figure 3. This is a direct consequence of the construction wherein the very small line segments were treated in just the same way as the original one but at a smaller size. An object which has such irregularity at arbitrarily small scales is said to have a *fine structure*. This is very different from, say, a circle where a small portion of the perimeter sufficiently magnifie will appear almost indistinguishable from a straight line.

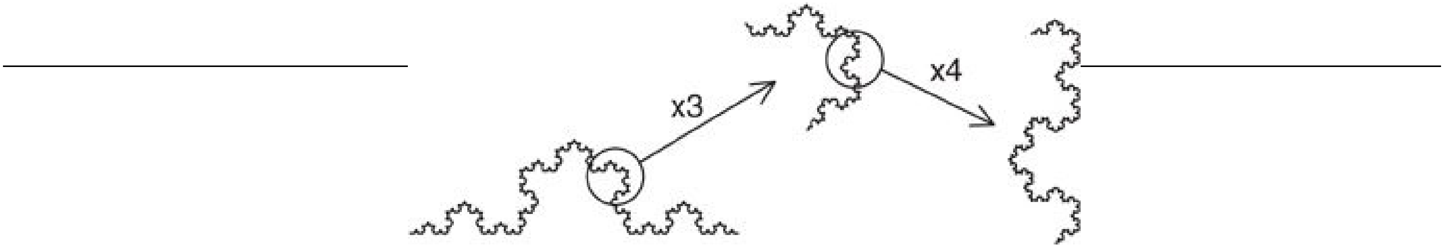


1. The basic step in constructing the von Koch curve: (a) divide a line into three equal parts, (b) remove the middle part, (c) replace the missing part by the other two sides of an equilateral triangle



2. (a) The first few stages E_0, E_1, E_2, … of construction of the von Koch curve *F*, (b) three such curves joined together to form the von Koch snowflake

Looking from another view point, the von Koch curve is *self-similar*, that is it is made up of smaller scale copies of itself. In particular we could cut it into 4 parts each a $\frac{1}{3}$ scale copy of the entire curve. Thinking of this another way, 4 photocopies of the whole von Koch curve at $33\frac{1}{3}$ per cent reduction can be positioned and joined together to form a curve identical to the original. But there are also self-similarities at smaller scales: the curve is made up of 16 copies of itself at $\frac{1}{9}$ scale, 64 copies at $\frac{1}{27}$ scale and so on—everywhere on the curve and at whatever enlargement there are many tiny copies of the whole curve.



3. Repeated magnification reveals more irregularity

Smooth curves—circles, ellipses, etc.—have a tangent at each point, that is a straight line that neatly touches the curve. This can be thought of as the instantaneous direction of travel of a point traversing the curve. This notion of a tangent or direction at each point is central to way that mathematicians have for centuries studied curves (it is the essence of the 'calculus'). However, the von Koch curve does not have a well-defined direction or slope at any point and one cannot draw tangents to the curv It is far too irregular to be described in traditional geometrical language and, unlike classical shapes, cannot be expressed by a 'simple' formula—*the methods of classical mathematics are not applicable* to the von Koch curve.

To find the perimeter length of a circle one might 'walk' around the circle taking very small steps, an multiply the step length by the number of steps taken. If the circle is of radius 1 (kilometres, metres, yards—the units don't matter so long as we are consistent), then if the steps are small the answer wil be very close to 6.283 (that is 2π) which is the perimeter length. We might try the same approach to measure the length of the von Koch curve. Assuming that the initial line segment in the construction has length 1 then traversing the curve with a step of length $1/10$ will require about 19 steps, corresponding to a length walked of $19 \times 1/10 = 1.9.$ If we reduce our step lengths to $1/100$ then our walk around the curve will visit many more of the small 'corners' of the curve and will take about 33 steps, a length $334 \times 1/100 = 3.34.$ And if we take tiny steps of, say, length $1/1{,}000{,}000$ (one-millionth) following around the irregularities of the curve will necessitate about 37.25 million steps, giving a length of 37.25. Unlike with the circle, trying to measure the von Koch curve by dividing it into shorter and shorter steps just gives ever larger estimates for its length. *The size depends on the scale at which the length is measured*—another property that distinguishes the von Koch curve from the circle or other classical geometrical figures.

In all these ways the von Koch curve is a very intricate and complicated object. Yet, in other ways it very simple. Its construction can be described in a short sentence: 'Repeatedly replace the middle third of each line segment with the other two sides of an equilateral triangle.' It is a *recursive construction*—that is it involves performing a simple step over and over again.

If we take three copies of the von Koch curve and join them corner to corner in a 'triangle', we get a rather idealized 'von Koch snowflake', shown in Figure 2(b)—with a bit of imagination the curve has a *natural appearance*.

To summarize, the von Koch curve possesses the following properties:

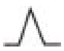
- *Fine structure*—detail at all scales, however small
- *Self-similarity*—made up of small scale copies of itself in some way

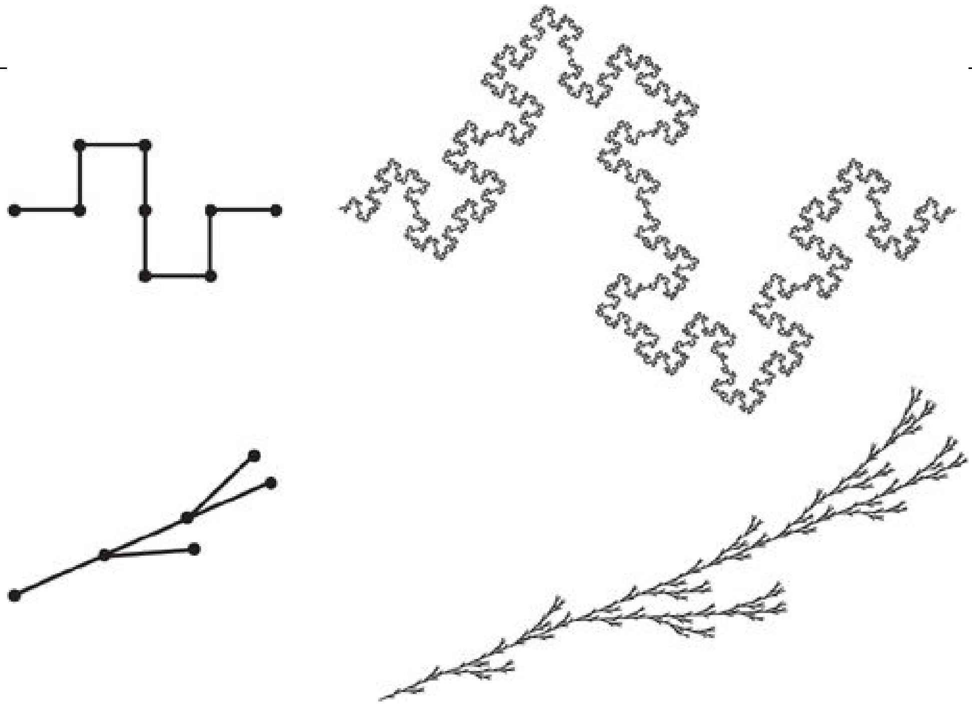
- *Classical methods of geometry and mathematics are not applicable*
- *‘Size’ depends on the scale at which it is measured*
- *A simple recursive construction*
- *A natural appearance*

A curve or other object with such properties is called a *fractal*, a word coined by Benoit Mandelbrot i 1975 from the Latin *fractus*, meaning ‘broken’. These features will be prominent in the wide range o fractals that will be encountered in the following pages.

A word of caution about this ‘definition’ of a fractal. We have termed an object a fractal if it satisfie several conditions, some of which are a little vaguely framed. Is there a more precise definition of a fractal? There has been considerable debate about this ever since the term was introduced. Mandelbr originally proposed a technical definition in terms of dimensions, but this was abandoned because there were plenty of objects that did not fit in with the definition but which clearly ought to be considered fractals. The current consensus is to regard something as a fractal if all or most of the properties listed above (along with one or two other more technical ones) hold in some form. This is somewhat analogous to the way biologists define ‘life’. Something is held to be alive if it has all or most of the characteristics on a list: ability to grow, ability to reproduce, ability to respond to stimuli in some way, etc. Nevertheless, there are things which are obviously ‘alive’ but which don’t have quite all the properties on the list.

Some more examples

Fractals with a completely different appearance may be constructed by a very similar recursive procedure. We obtained the von Koch curve by repeatedly replacing each straight line segment by a simple figure  sometimes called a *generator* or *motif* of the curve. Figure 4 shows some fractals constructed using other generators, again by replacing line segments by scaled down copies of the generator. The first example is a ‘squig’ curve generated by a ‘square wave’. The second is a fractal ‘grass’ with its natural appearance a consequence of the ‘twigs’ of the generator being at a slight angle to the main ‘stem’. Note how the list of fractal properties applies in each case. There are endless possibilities for producing fractals in this way from different generators.



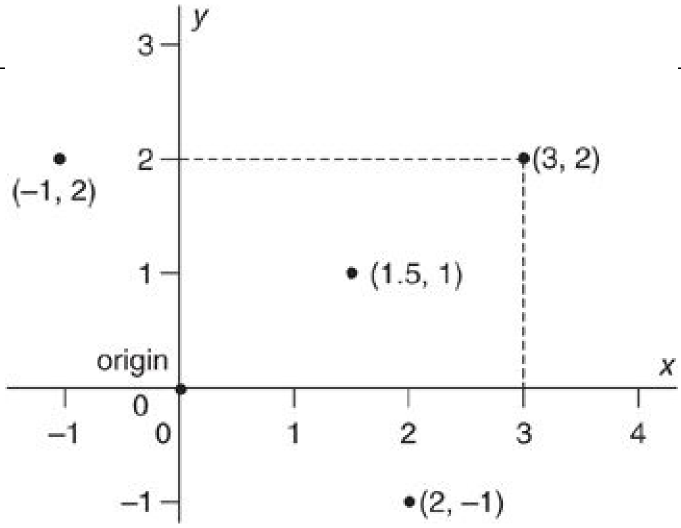
4. A squig curve (above) and a fractal grass (below) with their generators: the line segments in the generators are repeatedly replaced by scaled down copies to form the fractals

Coordinates, functions, and itineraries

The fractals we have seen are drawn on a piece of paper and could equally well be displayed on a computer screen. Such a flat surface is referred to as a *plane*. Of central interest to us are black and white 'pictures' drawn on a plane, that is any collection of points on the plane regarded as an entity. We will refer to such a collection of points as an *object*, a *shape*, a *figure*, or, to use the mathematician's term, a *set*. Thus, in general, a 'set' might refer to a circle, a von Koch curve, a collection of paint splashes on a page, or the silhouette of a person.

To describe fractals, and indeed any set, we need a means of specifying positions on a plane. The standard way of doing this uses coordinates, sometimes called Cartesian coordinates after René Descartes (1596–1650) who introduced them, which specify points in a similar way to longitude and latitude on a map, or the eastings and northings of a map grid reference.

Choose some point of the plane, called the *origin*. Any point of the plane may be reached from the origin by travelling a certain distance horizontally and then turning through a right angle and travelling another distance vertically. The location of the point is given by a pair of numbers, the firs being the horizontal distance travelled and the second the vertical distance. These numbers are the *coordinates* of the point. Thus the pair of coordinates (3, 2) gives the position of the point 'distance 3 along and 2 up', see Figure 5. It is usual to draw horizontal and vertical lines through the origin, calle the *x-axis* and *y-axis*, respectively, which may be marked with scales that indicate the distances in each of the perpendicular directions. By convention, positive numbers represent distances to the righ or upwards from the origin, and negative numbers denote distances to the left or downwards. We usually identify the coordinate pair with the point itself, so we refer to 'the point (3, 2) in the plane'.



5. Points in the plane with their coordinates

We also need to relate points in the plane, and one very useful way of doing this is using ‘functions’. For our purposes, a *function* is simply a rule or formula which, for each point in the plane, specifies some other point. We will frequently think of a function as an instruction that tells you how to move around the plane: if you are at some point, the function tells you another location to which to move. (Technically, such a function should be referred to as a ‘function from the plane to itself’, but all functions referred to in this book will be of this form.) We will use arrows to denote functions, for example, the function

$$(x,y) \to (x+1,y) \qquad (1)$$

says that if you are at the point with coodinates (x, y) you move to the point with coordinates (x + 1, y). The effect of this function on a specific point, (2, 5), say, is found by substituting the pair (2, 5) into the formula (1) replacing x by 2 and y by 5, so that (2,5) → (2 + 1, 5) = (3, 5); thus the function moves the point (2,5) to (3,5). Similarly, it moves the point (3, 7) to (4, 7) and (–3, –7) to (–2, –7). This function has the effect of adding 1 to the first coordinate of a point and leaving the second coordinate unchanged, so geometrically the function moves every point a distance 1 to the right.

For another example, the function given by

$$(x,y) \to (\tfrac{1}{2}x, \tfrac{1}{2}y) \qquad (2)$$

halves both coordinates, so takes each point to the point midway between it and the origin, so, for example, (6, 2) → (3, 1) and (3, 2) → (1.5, 1), see Figure 5.

We saw with the von Koch curve that repeating a simple operation over and over again produced an intricate fractal. Similarly, applying a function repeatedly can also lead to highly complex objects. Given a function and some initial point in the plane the function tells us to move to some new point. Applying the function to this second point takes us to a third point, applying the function to this third point takes us to a fourth point, and so on. Repeatedly applying the function takes us on a tour or *itinerary* around the plane, visiting a sequence of points in turn. This is rather like a treasure hunt of the type popular at children’s parties. An initial clue tells a child to go to some location where another clue may be found. This directs them to a further clue, and so on, so that they visit an itinerary of locations around a garden or park. A function is just a concise way of expressing such ‘clues’—