

ELEMENTARY AND MIDDLE SCHOOL  
MATHEMATICS

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*Teaching Developmentally*

EIGHTH EDITION



JOHN A. VAN DE WALLE  
KAREN S. KARP JENNIFER M. BAY-WILLIAMS

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# **Elementary and Middle School Mathematics**

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E I G H T H   E D I T I O N

# Elementary and Middle School Mathematics

*Teaching Developmentally*

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# Brief Contents



## SECTION I

### Teaching Mathematics: Foundations and Perspectives

<b>CHAPTER 1</b> Teaching Mathematics in the 21st Century	1	<b>CHAPTER 5</b> Building Assessment into Instruction	78
<b>CHAPTER 2</b> Exploring What It Means to Know and Do Mathematics	13	<b>CHAPTER 6</b> Teaching Mathematics Equitably to All Children	94
<b>CHAPTER 3</b> Teaching Through Problem Solving	32	<b>CHAPTER 7</b> Using Technological Tools to Teach Mathematics	113
<b>CHAPTER 4</b> Planning in the Problem-Based Classroom	59		

## SECTION II

### Development of Mathematical Concepts and Procedures

<b>CHAPTER 8</b> Developing Early Number Concepts and Number Sense	128	<b>CHAPTER 18</b> Proportional Reasoning	357
<b>CHAPTER 9</b> Developing Meanings for the Operations	148	<b>CHAPTER 19</b> Developing Measurement Concepts	375
<b>CHAPTER 10</b> Helping Students Master the Basic Facts	171	<b>CHAPTER 20</b> Geometric Thinking and Geometric Concepts	402
<b>CHAPTER 11</b> Developing Whole-Number Place-Value Concepts	192	<b>CHAPTER 21</b> Developing Concepts of Data Analysis	434
<b>CHAPTER 12</b> Developing Strategies for Addition and Subtraction Computation	216	<b>CHAPTER 22</b> Exploring Concepts of Probability	454
<b>CHAPTER 13</b> Developing Strategies for Multiplication and Division Computation	236	<b>CHAPTER 23</b> Developing Concepts of Exponents, Integers, and Real Numbers	472
<b>CHAPTER 14</b> Algebraic Thinking: Generalizations, Patterns, and Functions	258	<b>APPENDIX A</b> Standards for Mathematical Practice	491
<b>CHAPTER 15</b> Developing Fraction Concepts	290	<b>APPENDIX B</b> Standards for Teaching Mathematics	493
<b>CHAPTER 16</b> Developing Strategies for Fraction Computation	315	<b>APPENDIX C</b> Guide to Blackline Masters	495
<b>CHAPTER 17</b> Developing Concepts of Decimals and Percents	338		



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# Detailed Contents



Preface xix

## SECTION I

### Teaching Mathematics: Foundations and Perspectives

The fundamental core of effective teaching of mathematics combines an understanding of how children learn, how to promote that learning by teaching through problem solving, and how to plan for and assess that learning on a daily basis. Introductory chapters in this section provide perspectives on trends in mathematics education and the process of doing mathematics. These chapters develop the core ideas of learning, teaching, planning, and assessment. Additional perspectives on mathematics for children with diverse backgrounds and the role of technology are also discussed.



#### CHAPTER 1

### Teaching Mathematics in the 21st Century

1

- The National Standards-Based Movement** 1
- Principles and Standards for School Mathematics*** 2
  - The Six Principles 2
  - The Five Content Standards 3
  - The Five Process Standards 3
- Curriculum Focal Points: A Quest for Coherence*** 5
- Common Core State Standards*** 5
- Professional Standards for Teaching Mathematics and Mathematics Teaching Today*** 7
- Influences and Pressures on Mathematics Teaching** 7
  - National and International Studies 7
  - Curriculum 8
  - A Changing World Economy 8
- An Invitation to Learn and Grow** 9
  - Becoming a Teacher of Mathematics 9

#### RESOURCES FOR CHAPTER 1

- Recommended Readings 11
- Online Resources 11

#### REFLECTIONS ON CHAPTER 1

- Writing to Learn 11
- For Discussion and Exploration 12
- Field Experience Guide Connections 12



#### CHAPTER 2

### Exploring What It Means to Know and Do Mathematics

13

- What Does It Mean to Do Mathematics?** 13
    - Mathematics Is the Science of Pattern and Order 13
    - A Classroom Environment for Doing Mathematics 14
  - An Invitation to Do Mathematics** 15
    - Problems 15
    - Where Are the Answers? 19
  - What Does It Mean to Learn Mathematics?** 19
    - Constructivism 19
    - Sociocultural Theory 20
    - Implications for Teaching Mathematics 21
  - What Does It Mean to Understand Mathematics?** 23
    - Relational Understanding 24
    - Mathematics Proficiency 26
    - Benefits of Developing Mathematical Proficiency 28
  - Connecting the Dots** 29
- #### RESOURCES FOR CHAPTER 2
- Recommended Readings 30
  - Online Resources 30
- #### REFLECTIONS ON CHAPTER 2
- Writing to Learn 30
  - For Discussion and Exploration 30
  - Field Experience Guide Connections 31



## CHAPTER 3

### Teaching Through Problem Solving 32

#### A Shift in the Role of Problems 32

#### Teaching about Problem Solving 33

Four-Step Problem-Solving Process 33

Problem-Solving Strategies 34

#### Teaching Through Problem Solving 34

What Is a Problem? 34

Features of a Problem 35

Examples of Problems 35

#### Selecting Worthwhile Tasks 36

Levels of Cognitive Demand 36

Multiple Entry and Exit Points 37

Relevant Contexts 38

Using Textbooks 40

#### Orchestrating Classroom Discourse 42

Classroom Discussions 42

Questioning Considerations 44

Metacognition 45

How Much to Tell and Not to Tell 45

#### Writing to Learn 46

#### Equity and Teaching Through Problem Solving 47

#### A Three-Phase Lesson Format 49

The *Before* Phase of a Lesson 49

Teacher Actions in the *Before* Phase 49

The *During* Phase of a Lesson 52

Teacher Actions in the *During* Phase 52

The *After* Phase of a Lesson 54

Teacher Actions in the *After* Phase 54

#### Frequently Asked Questions 55

#### RESOURCES FOR CHAPTER 3

Recommended Readings 56

Online Resources 57

#### REFLECTIONS ON CHAPTER 3

Writing to Learn 57

For Discussion and Exploration 57

Field Experience Guide Connections 58



## CHAPTER 4

### Planning in the Problem-Based Classroom 59

#### Planning a Problem-Based Lesson 59

Planning Process for Developing a Lesson 59

Applying the Planning Process 63

Applying the Three-Phase Model to Short Tasks 63

Textbooks as Resources 64

#### Planning for All Learners 64

Make Accommodations and Modifications 64

Differentiating Instruction 65

Flexible Grouping 67

English Language Learners 68

Students with Special Needs 68

#### Drill or Practice? 71

New Definitions of Drill and Practice 71

What Practice Provides 71

What Drill Provides 71

When Is Drill Appropriate? 72

Drill and Student Misconceptions 72

#### Homework and Parental Involvement 72

Effective Homework 72

Beyond Homework: Families Doing Math 73

Resources for Families 73

#### RESOURCES FOR CHAPTER 4

Recommended Readings 74

Online Resources 74

#### REFLECTIONS ON CHAPTER 4

Writing to Learn 74

For Discussion and Exploration 75

Field Experience Guide Connections 75

#### EXPANDED LESSON Fixed Areas 76



## CHAPTER 5

### Building Assessment into Instruction 78

#### Integrating Assessment into Instruction 78

What Is Assessment? 78

The *Assessment Standards* 79

Why Do We Assess? 79

What Should Be Assessed? 80

#### Performance-Based Tasks 81

Examples of Performance-Based Tasks 81

Public Discussion of Performance Tasks 82

Rubrics and Performance Indicators 82

#### Writing and Journals 84

Journals 84

Writing Prompts 85

Writing for Early Learners 85

#### Student Self-Assessment 86

#### Tests 87

**Observations** 88  
 Anecdotal Notes 88  
 Rubrics 89  
 Checklists 89

**Diagnostic Interviews** 90

**Summative Assessments: Improving Performance on High-Stakes Tests** 90

**Using Assessments to Grade** 91  
 What Gets Graded Gets Valued 91  
 From Assessment Tools to Grades 91

**Using Assessments to Shape Instruction** 92

**RESOURCES FOR CHAPTER 5**

Recommended Readings 92  
 Online Resources 92

**REFLECTIONS ON CHAPTER 5**

Writing to Learn 93  
 For Discussion and Exploration 93  
 Field Experience Guide Connections 93



**CHAPTER 6**

**Teaching Mathematics Equitably to All Children** 94

**Mathematics for All Students** 94  
 Tracking Versus Differentiation 95  
 Instructional Principles for Diverse Learners 95

**Providing for Students Who Struggle and Those with Special Needs** 96  
 Prevention Models and Interventions for All Students 96  
 Students with Mild Disabilities 98  
 Students with Moderate/Severe Disabilities 100

**Students Who Are Culturally and Ethnically Diverse** 102

**Students Who Are English Language Learners (ELLs)** 104

**Gender-Friendly Mathematics Classrooms** 107  
 Possible Causes of Gender Differences 107  
 What Can We Try? 108

**Reducing Resistance and Building Resilience in Students with Low Motivation** 108

**Students Who Are Mathematically Gifted** 109

**Final Thoughts** 111

**RESOURCES FOR CHAPTER 6**

Recommended Readings 111  
 Online Resources 111

**REFLECTIONS ON CHAPTER 6**

Writing to Learn 112  
 For Discussion and Exploration 112  
 Field Experience Guide Connections 112



**CHAPTER 7**

**Using Technological Tools to Teach Mathematics** 113

**Technology-Supported Learning Activities** 114

**Calculators in Mathematics Instruction** 115  
 When to Use a Calculator 116  
 Benefits of Calculator Use 116  
 Graphing Calculators 117  
 Portable Data-Collection Devices 118

**Digital Tools in Mathematics Instruction** 119  
 Tools for Developing Numeration 119  
 Tools for Developing Geometry 120  
 Tools for Developing Probability and Data Analysis 121  
 Tools for Developing Algebraic Thinking 121

**Instructional Applications** 122  
 Concept Instruction 122  
 Problem Solving 122  
 Drill and Reinforcement 122

**Guidelines for Selecting and Using Digital Resources** 122  
 Guidelines for Using Digital Content 123  
 How to Select Appropriate Digital Content 123

**Resources on the Internet** 124  
 How to Select Internet Resources 124  
 Emerging Technologies 124

**RESOURCES FOR CHAPTER 7**

Recommended Readings 126  
 Online Resources 126

**REFLECTIONS ON CHAPTER 7**

Writing to Learn 127  
 For Discussion and Exploration 127  
 Field Experience Guide Connections 127

**SECTION II****Development of Mathematical Concepts and Procedures**

This section serves as the application of the core ideas of Section I. Here you will find chapters on every major content area in the pre-K–8 mathematics curriculum. Numerous problem-based activities to engage students are interwoven with a discussion of the mathematical content and how children develop their understanding of that content. At the outset of each chapter, you will find a listing of “Big Ideas,” the mathematical umbrella for the chapter. Also included are ideas for incorporating children’s literature, technology, and assessment. These chapters are designed to help you develop pedagogical strategies and to serve as a resource for your teaching now and in the future.

**CHAPTER 8****Developing Early Number Concepts and Number Sense 128****Promoting Good Beginnings 128****The Number Core: Quantity, Counting, and Knowing How Many 129**

- Quantity and the Ability to Subitize 129
- Early Counting 130
- Numeral Writing and Recognition 132
- Counting On and Counting Back 133

**The Relations Core: More Than, Less Than, and Equal To 134****Early Number Sense 135****Relationships Between Numbers 1 Through 10 136**

- One and Two More, One and Two Less 136
- Anchoring Numbers to 5 and 10 137
- Part-Part-Whole Relationships 139
- Dot Cards as a Model for Teaching Number Relationships 142

**Relationships for Numbers 10 Through 20 143**

- Pre-Place-Value Concepts 143
- Extending More Than and Less Than Relationships 144

**Number Sense in Their World 144**

- Calendar Activities 144
- Estimation and Measurement 145
- Data Collection and Analysis 145

**RESOURCES FOR CHAPTER 8**

- Literature Connections 146
- Recommended Readings 146
- Online Resources 146

**REFLECTIONS ON CHAPTER 8**

- Writing to Learn 147
- For Discussion and Exploration 147
- Field Experience Guide Connections 147

**CHAPTER 9****Developing Meanings for the Operations 148****Addition and Subtraction Problem Structures 149**

- Examples of Change Problems 149
- Examples of Part-Part-Whole Problems 150
- Examples of Compare Problems 150
- Problem Difficulty 151
- Computational and Semantic Forms of Equations 151

**Teaching Addition and Subtraction 151**

- Contextual Problems 152

**INVESTIGATIONS IN NUMBER, DATA, AND SPACE****Grade 2, Counting, Coins, and Combinations 153**

- Model-Based Problems 154
- Properties of Addition and Subtraction 157

**Multiplication and Division Problem Structures 158**

- Examples of Equal-Group Problems 158
- Examples of Comparison Problems 159
- Examples of Combination Problems 159
- Examples of Area and Other Product-of-Measures Problems 160

**Teaching Multiplication and Division 160**

- Contextual Problems 160
- Remainders 161
- Model-Based Problems 162
- Properties of Multiplication and Division 164

**Strategies for Solving Contextual Problems 165**

- Analyzing Context Problems 165
- Two-Step Problems 167

**RESOURCES FOR CHAPTER 9**

- Literature Connections 168
- Recommended Readings 168
- Online Resources 169

**REFLECTIONS ON CHAPTER 9**

- Writing to Learn 169
- For Discussion and Exploration 169
- Field Experience Guide Connections 170

**CHAPTER 10****Helping Students Master the Basic Facts 171****Developmental Nature of Basic Fact Mastery 171**

- Approaches to Fact Mastery 172
- Guiding Strategy Development 173

**Reasoning Strategies for Addition Facts 174**

- One More Than and Two More Than 175
- Adding Zero 176
- Using 5 as an Anchor 176
- Make 10 176
- Up Over 10 176
- Doubles 177
- Near-Doubles 178
- Reinforcing Reasoning Strategies 178

**Reasoning Strategies for Subtraction Facts 179**

- Subtraction as Think-Addition 179
- Down Over 10 180
- Take from the 10 180

**Reasoning Strategies for Multiplication and Division Facts 181**

- Doubles 181
- Fives 181
- Zeros and Ones 181
- Nifty Nines 182
- Using Known Facts to Derive Other Facts 182
- Division Facts 183

**Mastering the Basic Facts 183**

- Effective Drill 184
- Games to Support Basic Fact Mastery 184

**Fact Remediation 186**

- What to Do When Teaching Basic Facts 188
- What Not to Do When Teaching Basic Facts 188

**RESOURCES FOR CHAPTER 10**

- Literature Connections 189
- Recommended Readings 189
- Online Resources 189

**REFLECTIONS ON CHAPTER 10**

- Writing to Learn 190
- For Discussion and Exploration 190
- Field Experience Guide Connections 191

**CHAPTER 11****Developing Whole-Number Place-Value Concepts 192****Pre-Base-Ten Understandings 192**

- Counting by Ones 193

**Basic Ideas of Place Value 193**

- Integration of Base-Ten Groupings with Counting by Ones 193
- Role of Counting 194
- Integration of Groupings with Words 194
- Integration of Groupings with Place-Value Notation 194

**Base-Ten Models for Place Value 195**

- Groupable Models 195
- Pregrouped or Trading Models 195
- Nonproportional Models 197

**Developing Base-Ten Concepts 197**

- Grouping Activities 197
- The Strangeness of *Ones*, *Tens*, and *Hundreds* 199
- Grouping Tens to Make 100 199
- Equivalent Representations 199

**Oral and Written Names for Numbers 200**

- Two-Digit Number Names 200
- Three-Digit Number Names 202
- Written Symbols 202
- Assessing Place-Value Concepts 203

**Patterns and Relationships with Multidigit Numbers 204**

- The Hundreds Chart 204
- Relationships with Landmark Numbers 206
- Connecting Place Value to Addition and Subtraction 206
- Connections to Real-World Ideas 211

**Numbers Beyond 1000 211**

- Extending the Place-Value System 211
- Conceptualizing Large Numbers 212

**RESOURCES FOR CHAPTER 11**

- Literature Connections 214
- Recommended Readings 214
- Online Resources 214

**REFLECTIONS ON CHAPTER 11**

- Writing to Learn 215
- For Discussion and Exploration 215
- Field Experience Guide Connections 215

**CHAPTER 12****Developing Strategies for Addition and Subtraction Computation 216****Toward Computational Fluency 217**

- Direct Modeling 217
- Student-Invented Strategies 218
- Standard Algorithms 219

**Development of Student-Invented Strategies 220**

- Creating an Environment for Inventing Strategies 221
- Models to Support Invented Strategies 221

**Student-Invented Strategies for Addition and Subtraction** 222

- Adding and Subtracting Single-Digit Numbers 222
- Adding Two-Digit Numbers 223
- Subtracting by Counting Up 224
- Take-Away Subtraction 224
- Extensions and Challenges 225

**Standard Algorithms for Addition and Subtraction** 226

- Standard Algorithm for Addition 226
- Standard Algorithm for Subtraction 227

**Introducing Computational Estimation** 228

- Understanding Computational Estimation 229
- Suggestions for Teaching Computational Estimation 229

**Computational Estimation Strategies** 231

- Front-End Methods 231
- Rounding Methods 231
- Compatible Numbers 231

**RESOURCES FOR CHAPTER 12**

- Literature Connections 233
- Recommended Readings 233
- Online Resources 233

**REFLECTIONS ON CHAPTER 12**

- Writing to Learn 234
- For Discussion and Exploration 234
- Field Experience Guide Connections 235



**CHAPTER 13**

**Developing Strategies for Multiplication and Division Computation** 236

**Student-Invented Strategies for Multiplication** 237

- Useful Representations 237
- Multiplication by a Single-Digit Multiplier 237
- Multiplication of Larger Numbers 239

**Standard Algorithm for Multiplication** 241

- One-Digit Multipliers 241
- Two-Digit Multipliers 242

**Student-Invented Strategies for Division** 243

- Missing-Factor Strategies 244
- Cluster Problems 244

**Standard Algorithm for Division** 245

- One-Digit Divisors 245
- Two-Digit Divisors 246

**Computational Estimation in Multiplication and Division** 249

- Understanding Computational Estimation 249
- Suggestions for Teaching Computational Estimation 249

**Computational Estimation from Invented Strategies** 250

- Stop Before the Details 250
- Use Related Problem Sets 251

**Computational Estimation Strategies** 251

- Front-End Methods 251

- Rounding Methods 252
- Compatible Numbers 253
- Using Tens and Hundreds 253

**Estimation Experiences** 253

- Calculator Activities 254
- Using Whole Numbers to Estimate Rational Numbers 255

**RESOURCES FOR CHAPTER 13**

- Literature Connections 255
- Recommended Readings 256
- Online Resources 256

**REFLECTIONS ON CHAPTER 13**

- Writing to Learn 256
- For Discussion and Exploration 256
- Field Experience Guide Connections 257



**CHAPTER 14**

**Algebraic Thinking: Generalizations, Patterns, and Functions** 258

**Algebraic Thinking** 259

**Generalization from Arithmetic** 259

- Generalization with Operations 259
- Generalization in the Hundreds Chart 260
- Generalization Through Exploring a Pattern 261

**Meaningful Use of Symbols** 262

- The Meaning of the Equal Sign 262
- The Meaning of Variables 266

**Making Structure in the Number System Explicit** 270

- Making Conjectures about Properties 270
- Justifying Conjectures 271
- Odd and Even Relationships 272

**Study of Patterns and Functions** 272

- Repeating Patterns 272
- Growing Patterns 273
- Linear Functions 279

**Mathematical Modeling** 280

**Teaching Considerations** 281

- Emphasize Appropriate Algebra Vocabulary 281
- Connecting Representations 282
- Algebraic Thinking Across the Curriculum 284

**CONNECTED MATHEMATICS** *Grade 7, Variables and Patterns* 285

**RESOURCES FOR CHAPTER 14**

- Literature Connections 287
- Recommended Readings 287
- Online Resources 288

**REFLECTIONS ON CHAPTER 14**

- Writing to Learn 288
- For Discussion and Exploration 288
- Field Experience Guide Connections 289



**CHAPTER 15**

**Developing Fraction Concepts 290**

**Meanings of Fractions 290**

- Fraction Constructs 291
- Why Fractions Are So Difficult 291

**Models for Fractions 292**

- Area Models 293
- Length Models 293
- Set Models 295

**Concept of Fractional Parts 295**

- Fraction Size Is Relative 295
- Fraction Language 296
- Partitioning 296
- Sharing Tasks 298
- Iterating 299
- Fraction Notation 302
- Fractions Greater Than 1 302

**Estimating with Fractions 303**

**Equivalent Fractions 304**

- Conceptual Focus on Equivalence 304
- Equivalent-Fraction Models 305
- Developing an Equivalent-Fraction Algorithm 308

**Comparing Fractions 310**

- Comparing Fractions Using Number Sense 310
- Using Equivalent Fractions to Compare 311

**Teaching Considerations for Fraction Concepts 312**

**RESOURCES FOR CHAPTER 15**

- Literature Connections 312
- Recommended Readings 313
- Online Resources 313

**REFLECTIONS ON CHAPTER 15**

- Writing to Learn 313
- For Discussion and Exploration 313
- Field Experience Guide Connections 314



**CHAPTER 16**

**Developing Strategies for Fraction Computation 315**

**Understanding Fraction Operations 315**

- Conceptual Development Takes Time 316
- A Problem-Based Number-Sense Approach 316
- Computational Estimation 317

**Addition and Subtraction 319**

- Contextual Examples and Invented Strategies 319
- Models 320
- Developing the Algorithms 322
- Fractions Greater Than One 324
- Addressing Misconceptions 324

**Multiplication 325**

- Contextual Examples and Models 325
- Developing the Algorithms 329
- Factors Greater Than One 329
- Addressing Misconceptions 330

**Division 331**

- Contextual Examples and Models 331
- Answers That Are Not Whole Numbers 333
- Developing the Algorithms 333
- Addressing Misconceptions 335

**RESOURCES FOR CHAPTER 16**

- Literature Connections 335
- Recommended Readings 335
- Online Resources 336

**REFLECTIONS ON CHAPTER 16**

- Writing to Learn 336
- For Discussion and Exploration 336
- Field Experience Guide Connections 337



**CHAPTER 17**

**Developing Concepts of Decimals and Percents 338**

**Extending the Place-Value System 338**

**Connecting Fractions and Decimals 341**

- Base-Ten Fractions 341

**Developing Decimal Number Sense 344**

- Familiar Fractions Connected to Decimals 344

**Computation with Decimals 348**

- The Role of Estimation 348
- Addition and Subtraction 349
- Multiplication 349
- Division 350

**Introducing Percents 351**

- Models and Terminology 351
- Percent Problems in Context 353
- Estimation 354

**RESOURCES FOR CHAPTER 17**

- Literature Connections 355
- Recommended Readings 355
- Online Resources 355

**REFLECTIONS ON CHAPTER 17**

- Writing to Learn 356
- For Discussion and Exploration 356
- Field Experience Guide Connections 356



**CHAPTER 18****Proportional Reasoning 357****Ratios 357**

- Types of Ratios 357
- Ratios Compared to Fractions 358
- Two Ways to Think about Ratios 358

**Proportional Reasoning 359**

- Proportional and Nonproportional Situations 359
- Additive and Multiplicative Comparisons in Problems 360
- Covariation 362

**Develop a Wide Variety of Strategies 366**

- Mental Strategies 366

**CONNECTED MATHEMATICS Grade 7, Comparing and Scaling 368**

- Ratio Tables 369
- Double Line (Strip) Comparison 370
- Percents 370
- Cross-Products 371

**Teaching Proportional Reasoning 372****RESOURCES FOR CHAPTER 18**

- Literature Connections 372
- Recommended Readings 373
- Online Resources 373

**REFLECTIONS ON CHAPTER 18**

- Writing to Learn 374
- For Discussion and Exploration 374
- Field Experience Guide Connections 374

**CHAPTER 19****Developing Measurement Concepts 375****The Meaning and Process of Measuring 375**

- Concepts and Skills 376
- Introducing Nonstandard Units 377
- Developing Standard Units 378
- Instructional Goals 378
- Important Standard Units and Relationships 379
- The Role of Estimation and Approximation 379
- Strategies for Estimating Measurements 379
- Tips for Teaching Estimation 380
- Measurement Estimation Activities 381

**Length 381**

- Comparison Activities 381
- Using Models of Length Units 382
- Making and Using Rulers 383

**Area 384**

- Comparison Activities 384
- Using Models of Area Units 385

**INVESTIGATIONS IN NUMBER, DATA, AND SPACE****Grade 3, Perimeter, Angles, and Area 387**

- The Relationship Between Area and Perimeter 387

**Developing Formulas for Area 388**

- Student Misconceptions 388
- Areas of Rectangles, Parallelograms, Triangles, and Trapezoids 389
- Circumference and Area of Circles 391

**Volume and Capacity 391**

- Comparison Activities 391
- Using Models of Volume and Capacity Units 393
- Using Measuring Cups 393
- Developing Formulas for Volumes of Common Solid Shapes 393
- Connections Between Formulas 395

**Weight and Mass 395**

- Comparison Activities 395
- Using Models of Weight or Mass Units 395

**Angles 395**

- Comparison Activities 395
- Using Models of Angular Measure Units 395
- Using Protractors and Angle Rulers 396

**Time 396**

- Comparison Activities 396
- Reading Clocks 397
- Elapsed Time 398

**Money 399**

- Recognizing Coins and Identifying Their Values 399
- Counting Sets of Coins 399
- Making Change 399

**RESOURCES FOR CHAPTER 19**

- Literature Connections 399
- Recommended Readings 400
- Online Resources 400

**REFLECTIONS ON CHAPTER 19**

- Writing to Learn 400
- For Discussion and Exploration 400
- Field Experience Guide Connections 401

**CHAPTER 20****Geometric Thinking and Geometric Concepts 402****Geometry Goals for Students 402**

- Spatial Sense and Geometric Reasoning 403
- Geometric Content 403

**Developing Geometric Thinking 403**

- The van Hiele Levels of Geometric Thought 403
- Implications for Instruction 407

**Learning about Shapes and Properties 407**

- Shapes and Properties for Level-0 Thinkers 408

Shapes and Properties for Level-1 Thinkers 411  
 Shapes and Properties for Level-2 Thinkers 416

**Learning about Transformations** 419  
 Transformations for Level-0 Thinkers 419  
 Transformations for Level-1 Thinkers 421  
 Transformations for Level-2 Thinkers 423

**Learning about Location** 424  
 Location for Level-1 Thinkers 425  
 Location for Level-2 Thinkers 427

**Learning about Visualization** 428  
 Visualization for Level-0 Thinkers 428  
 Visualization for Level-1 Thinkers 429  
 Visualization for Level-2 Thinkers 430

**RESOURCES FOR CHAPTER 20**

Literature Connections 431  
 Recommended Readings 432  
 Online Resources 432

**REFLECTIONS ON CHAPTER 20**

Writing to Learn 432  
 For Discussion and Exploration 432  
 Field Experience Guide Connections 433



**CHAPTER 21**

**Developing Concepts of Data Analysis 434**

**What Does It Mean to Do Statistics?** 435  
 Is It Statistics or Is It Mathematics? 435  
 The Shape of Data 435  
 The Process of Doing Statistics 436

**Formulating Questions** 436  
 Classroom Questions 436  
 Beyond One Classroom 437

**Data Collection** 437  
 Collecting Data 437  
 Using Existing Data Sources 438

**Data Analysis: Classification** 438  
 Attribute Materials 438

**Data Analysis: Graphical Representations** 440  
 Bar Graphs and Tally Charts 441  
 Circle Graphs 442  
 Continuous Data Graphs 443  
 Scatter Plots 444

**Data Analysis: Measures of Center and Variability** 446  
 Averages 446  
 Understanding the Mean: Two Interpretations 446  
 Variability 449  
 Box Plots 450

**Interpreting Results** 451

**RESOURCES FOR CHAPTER 21**

Literature Connections 452  
 Recommended Readings 452  
 Online Resources 452

**REFLECTIONS ON CHAPTER 21**

Writing to Learn 453  
 For Discussion and Exploration 453  
 Field Experience Guide Connections 453



**CHAPTER 22**

**Exploring Concepts of Probability 454**

**Introducing Probability** 455  
 Likely or Not Likely 455  
 The Probability Continuum 457

**Theoretical Probability and Experiments** 459  
 Theoretical Probability 459  
 Experiments 460  
 Why Use Experiments? 462  
 Use of Technology in Experiments 463

**Sample Spaces and Probability of Two Events** 463  
 Independent Events 463  
 Area Models 465  
 Dependent Events 466

**Simulations** 467

**RESOURCES FOR CHAPTER 22**

Literature Connections 469  
 Recommended Readings 469  
 Online Resources 470

**REFLECTIONS ON CHAPTER 22**

Writing to Learn 470  
 For Discussion and Exploration 470  
 Field Experience Guide Connections 471



**CHAPTER 23**

**Developing Concepts of Exponents, Integers, and Real Numbers 472**

**Exponents** 472  
 Exponents in Expressions and Equations 473  
 Order of Operations 473  
 Negative Exponents 476  
 Scientific Notation 476

**Integers** 478  
 Contexts for Exploring Integers 478  
 Quantity Contexts 479  
 Linear Contexts 479

Meaning of Negative Numbers 480  
Models for Teaching Integers 481  
**Operations with Integers** 481  
Addition and Subtraction 481  
Multiplication and Division 484  
**Real Numbers** 486  
Rational Numbers 486  
Irrational Numbers 486

**RESOURCES FOR CHAPTER 23**

Literature Connections 488  
Recommended Readings 488  
Online Resources 488

**REFLECTIONS ON CHAPTER 23**

Writing to Learn 489  
For Discussion and Exploration 489  
Field Experience Guide Connections 489

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**APPENDIX A**

**Standards for Mathematical Practice 491**

**APPENDIX B**

**Standards for Teaching Mathematics 493**

**APPENDIX C**

**Guide to Blackline Masters 495**

**References 507**

**Index 527**

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# Preface



## NEW TO THIS EDITION

The eighth edition has been revised to include the following changes to better prepare teachers to teach mathematics to all learners:

- **New adaptations and accommodations for English language learners and students with disabilities** appear not only in the narrative in Section I but also in many activities through direct examples and descriptions for the various content areas in Section II. The increased emphasis on diversity will be obvious to those who have used the book in the past. Chapter 4 (Planning in the Problem-Based Classroom) has an increased focus on planning for all learners, including new coverage on considerations for students with disabilities to complement the revised section on ELLs. Chapter 6 (Teaching Mathematics Equitably to All Children) contains significant updates to each section and alignment with the research synthesis from *RtI Practice Guide for Students Struggling in Mathematics* (Gersten, Beckmann, Clarke, Foegen, Marsh, Star, & Witzel, 2009). New to this chapter are strategies for cultural/ethnic differences, including Table 6.4, “Reflective Questions to Focus on Culturally Responsive Mathematics Instruction,” and additional guidance for teachers on students with disabilities, including a chart of common stumbling blocks. Importantly, in Section II the Activities feature specific adaptations and accommodations for ELLs and students with disabilities. These provide specific ways to make the activity accessible and still challenging.
- **Revised Expanded Lessons** located in the book, the *Field Experience Guide*, and on MyEducationLab ([www.myeducationlab.com](http://www.myeducationlab.com)) now include tips and strategies for English language learners and students with disabilities.
- **Increased emphasis on student misconceptions and how to address them effectively** will better support teachers’ understanding of what needs explicit attention when teaching mathematics. Since the publication of the seventh edition, an increasing body of research has emerged on students’ misconceptions and naïve understandings in a variety of mathematics content. Throughout Section II, the research about misconceptions and gaps in student mathematical knowledge is presented to assist teachers in the identification and preparation for these common barriers to understanding. In such topics as fractions and decimals, related findings allow teachers to plan ahead using examples and counterexamples to strengthen student understanding as they face what may be expected areas of confusion.
- **New samples of authentic student work** illustrate student thinking. Responses to problem-based assignments present glimpses into how students think about problems and what students’ written work on mathematical tasks looks like, increasing teachers’ awareness of how rich students’ mathematical thinking can be—and how high our expectations should be. Some student work also demonstrates naïve understandings.
- **Increased early childhood coverage** provides expanded emphasis on and reorganization of early numeracy in Chapters 8 and 9 reflecting the work of the Committee on Early Childhood Mathematics through the National Research Council. Based on learning trajectories and progressions for the core areas of number, relations, and operations, the work with early learners is seen as the essential foundation for number sense and problem solving.

- **New Formative Assessment Notes** in each chapter in Section II guide readers through ideas they can test with individual students or students in groups. Formative assessment is one of the key tools in finding out what students are thinking, and thereby identifying their areas of strength and weakness. Chapter 5, *Building Assessment into Instruction*, contains a more detailed description of formative assessments organized by Piaget's three major assessment areas: tasks, observation, and interviews. To bring these ideas to life and to make them more directly linked to the content, these Formative Assessment Notes are included throughout content area chapters to support teachers in the effective use of formative assessment, which is directly connected to increased student achievement.
- **New information on using NCTM standards and *Common Core State Standards*** to inform instruction appears in Chapter 1 and 2 and in relevant references in Section II. Not surprisingly, this book is aligned with the new *Common Core State Standards*, adopted by 44 of the 50 states at the time of publication. The *Common Core State Standards* and other standards documents are described in Chapter 1. The Standards for Mathematical Practice portion of the *Common Core State Standards* are addressed in Chapter 2 (connected to the *Adding it Up* mathematical proficiencies) and infused throughout the book. As essential content is described in Section II, specific standards are referenced, giving the appropriate grade level and treatment relevant to the content. In addition, chapter content has been adapted to reflect the attention given to the content in the *Common Core State Standards*. Appendix A provides the Standards for Mathematical Practice.
- **Extensively updated information on how to effectively integrate new technological tools** to support teaching and learning appears in Chapter 7 and throughout the text with marginal icons. Updated technology integration content and strategies now also appear in select Activities.
- **A reorganization of Chapters 12 and 13** emphasizes both strategies for computation and estimation for addition and subtraction in Chapter 12 and the same for multiplication and division in Chapter 13. This is a change from the seventh edition, which separated developing strategies for whole number computation and estimation for the four operations. Many reviewers suggested this change, infusing computational estimation in these new chapters, and this rearrangement links too to the developmental nature of those operations.
- **A discussion on engaging families in meaningful ways** to help students learn mathematics appears in Chapter 4.
- **Additional attention to classroom discourse** now appears in Chapter 3 (Teaching Through Problem Solving). The coverage includes how to conduct productive discussion sessions and develop effective questioning, and is illustrated with a vignette.

## OTHER CHANGES OF NOTE

Much has changed on the landscape of mathematics education, and so many aspects of the book have been updated to reflect those changes. In addition to the changes listed previously, the following substantive changes have been made:

- A new section on homework and parental involvement is provided in Chapter 4.
- There is an increased focus on the research-based three-phase developmental model of developing basic facts, and added new activities to support basic fact mastery appear in Chapter 10.
- The content on algebraic thinking has been adapted to align with current research and standards. Specifically, Kaput's five areas (from his 1999 work) are now three areas (based

on his 2008 work). Also, there is an increased emphasis on equivalence and variables, including adding the number-line representation of variables. Increased attention is given to making the properties (especially distributive) more explicit in response to the *Common Core State Standards*.

- Chapter 15 (Developing Fraction Concepts) has greatly expanded sections on partitioning and on equivalence to reflect three recent research reviews that have indicated that this is essential to all advanced fraction work and success in algebra.
- Chapter 16 (Developing Strategies for Fraction Computation) now includes Activities—ten new ideas for developing understanding of fraction operations.
- Chapter 18 has been shortened, had new activities added, and been refocused to address understanding of ratios more deeply (with less focus on connecting to other content areas).
- The chapter on measurement, Chapter 19, has been reorganized. Previously the development of all measurement formulas was shared at the end of the chapter; now the formulas are integrated with the corresponding measurement topic (e.g., area or volume).
- Chapter 21 gives more explicit attention to distinguishing between numerical data and categorical data.
- Chapter 23 includes a significantly revised section on order of operations and numerous new activities.

## WHAT YOU WILL FIND IN THIS BOOK

If you look at the table of contents, you will see that the chapters are separated into two distinct sections. The first section, consisting of seven chapters, deals with important ideas that cross the boundaries of specific areas of content. The second section, consisting of 16 chapters, offers teaching suggestions and activities for every major mathematics topic in the pre-K–8 curriculum. Chapters in Section I offer perspectives on the challenging task of helping students learn mathematics. Having a feel for the discipline of mathematics—that is, to know what it means to “do mathematics”—is critical to learning how to teach mathematics well. In addition, understanding constructivist and sociocultural perspectives on learning mathematics and how that is applied to teaching through problem solving provides a foundation and rationale for how to teach and assess pre-K–8 students.

Importantly, you will be teaching diverse students, including students who are English language learners, are gifted, or have disabilities. You will learn how to apply instructional strategies in ways that support and challenge *all* learners. Formative assessment strategies, strategies for diverse learners, and effective use of technological tools are addressed in specific chapters in Section I (Chapters 5, 6, and 7, respectively), and throughout Section II chapters.

Each chapter of Section II focuses on one of the major content areas in pre-K–8 mathematics curriculum. It begins with identifying the big ideas for that content, and also provides guidance on how students best learn that content and many problem-based activities to engage students in understanding mathematics. Reflecting on the activities as you read can help you think about the mathematics from the perspective of the student. As often as possible, take out pencil and paper and try the problems so that you actively engage in *your learning* about *students learning* mathematics. In so doing, we hope this book will increase your own understanding of mathematics, the students you teach, and how to teach them well.

## SOME SPECIAL FEATURES OF THIS TEXT

By flipping through the book, you will notice many section headings, a large number of figures, and various special features. All are designed to make the book more useful as a textbook and as a long-term resource. Here are a few things to look for.

### Big Ideas ▶

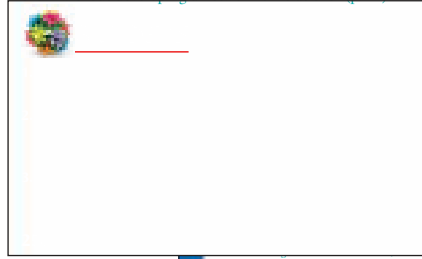
Much of the research and literature espousing a student-centered approach suggests that teachers plan their instruction around “big ideas” rather than isolated skills or concepts. At the beginning of each chapter in Section II, you will find a list of the key mathematical ideas associated with the chapter. Teachers find these lists helpful for quickly getting a picture of the mathematics they are teaching.

### Mathematics Content Connections ▶

Following the Big Ideas lists are brief descriptions of other content areas in mathematics that are related to the content of the current chapter. These lists are offered to help you be more aware of the potential interaction of content as you plan lessons, diagnose students’ difficulties, and learn more yourself about the mathematics you are teaching.

### Activities ▶

The numerous activities found in every chapter of Section II have always been rated by readers as one of the most valuable parts of the book. Some activity ideas are described directly in the text and in the illustrations. Others are presented in the numbered Activity boxes. Every activity is a problem-based task (as described in Chapter 3) and is designed to engage students in doing mathematics. New adaptations and accommodations for English language learners and students with disabilities are included in many activities.



◆ **Number, Place Value, Basic Facts, and Computation** (Chapters 8–13): The most important generalizations at the core of algebraic thinking are those made about number and computation—arithmetic. Not only does algebraic thinking generalize from number and computation, but also the generalizations themselves add to understanding and facility with computation. We can use our understanding of 10 to add  $5 + 8$  ( $5 + 8 = 3 + 2 + 8 = 3 + 10$ ) or  $5 + 38$  ( $5 + 38 = 3 + 2 + 38 = 3 + 40$ ). The generalized idea is that 2 can be taken from one addend and moved to the other:  $a + b = (a - 2) + (b + 2)$ . Although students may not symbolize this general idea, seeing that this regularly works is algebraic thinking. Making these regularities explicit supports students’ conceptual and procedural development of number as well as prepares them for the algebra they will explore in high school.

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◆ **Proportional Reasoning** (Chapter 18): Every proportional situation gives rise to a linear (straight-line) function with a graph

Activity 14.11 explores properties of odd and even

**Activity 14.11**

**Broken Calculator: Can You Fix It?**

Explore these two challenges; afterward ask students for conjectures they might make about odds and evens.

1. If you cannot use any of the even keys (0, 2, 4, 6, 8), can you create an even number in the calculator display? If so, how?
2. If you cannot use any of the odd keys (1, 3, 5, 7, 9), can you create an odd number in the calculator display? If so, how?

**Activity 14.12**

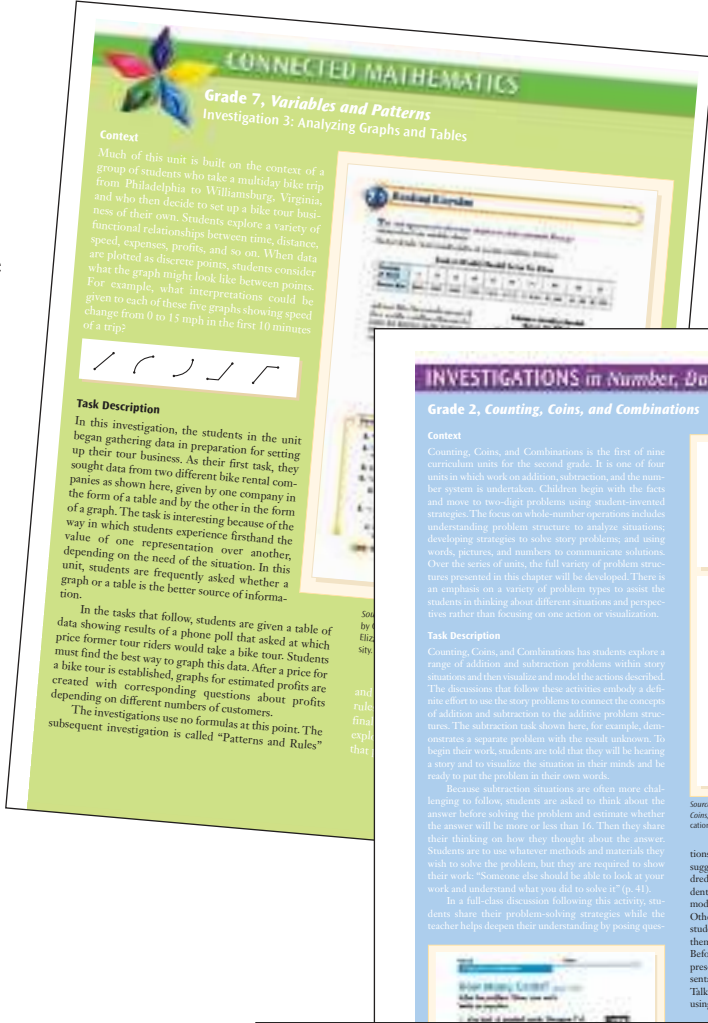
**Making Repeating Patterns**

Students can work independently or in groups of two or three to extend patterns made from simple materials: buttons, colored blocks, connecting cubes, toothpicks, geometric shapes—items you can gather easily. For each set of materials, draw or build two or three complete repetitions so the core is obvious. The students’ task is to extend it. Figure 14.10 illustrates one possible pattern for



## Investigations in Number, Data, and Space and Connected Mathematics

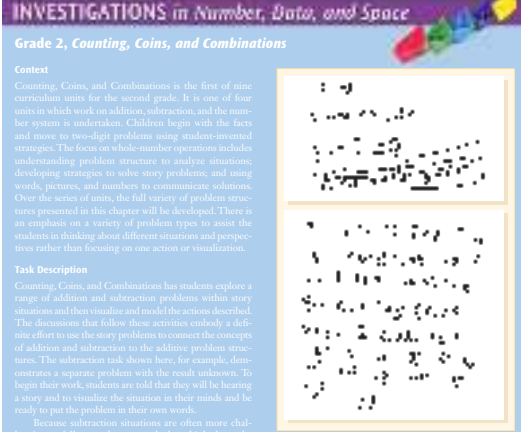
In Section II, four chapters include features that describe an activity from the standards-based curriculum *Investigations in Number, Data, and Space* (an elementary curriculum) or *Connected Mathematics Project (CMP II)* (a middle school curriculum). These features include a description of an activity in the program as well as the context of the unit in which it is found. The main purpose of this feature is to acquaint you with these materials and to demonstrate how the spirit of the NCTM *Standards* and the constructivist theory espoused in this book have been translated into existing commercial curricula.



**Context**  
Much of this unit is built on the context of a group of students who take a multi-day bike trip from Philadelphia to Williamsburg, Virginia, and who then decide to set up a bike tour business of their own. Students explore a variety of functional relationships between time, distance, speed, expenses, profits, and so on. When data are plotted as discrete points, students consider what the graph might look like between points. For example, what interpretations could be given to each of these five graphs showing speed change from 0 to 15 mph in the first 10 minutes of a trip?

**Task Description**  
In this investigation, the students in the unit began gathering data in preparation for setting up their tour business. As their first task, they sought data from two different bike rental companies as shown here, given by one company in the form of a table and by the other in the form of a graph. The task is interesting because of the value of one representation over another, depending on the need of the situation. In this unit, students are frequently asked whether a graph or a table is the better source of information.

In the tasks that follow, students are given a table of data showing results of a phone poll that asked at which price former tour riders would take a bike tour. Students must find the best way to graph this data. After a price for a bike tour is established, graphs for estimated profits are created with corresponding questions about profits depending on different numbers of customers. The investigations use no formulas at this point. The subsequent investigation is called "Patterns and Rules"



**Context**  
Counting, Coins, and Combinations is the first of nine curriculum units for the second grade. It is one of four units in which work on addition, subtraction, and the number system is undertaken. Children begin with the facts and move to two-digit problems using student-oriented strategies. The focus on whole-number operations includes understanding problem structure to analyze situations; developing strategies to solve story problems; and using words, pictures, and numbers to communicate solutions. Over the series of units, the full variety of problem structures presented in this chapter will be developed. There is an emphasis on a variety of problem types to assist the students in thinking about different situations and perspective rather than focusing on one action or visualization.

**Task Description**  
Counting, Coins, and Combinations has students explore a range of addition and subtraction problems within story situations and then visualize and model the actions described. The discussions that follow these activities embody a definite effort to use the story problems to connect the concepts of addition and subtraction to the additive problem structure. The subtraction task shown here, for example, demonstrates a separate problem with the results unknown. To begin their work, students are told that they will be hearing a story and to visualize the situation in their minds and be ready to put the problem in their own words. Because subtraction situations are often more challenging to follow, students are asked to think about the answer before solving the problem and estimate whether the answer will be more or less than 10. Then they share their thinking on how they thought about the answer. Students are to use whatever methods and materials they wish to solve the problem, but they are required to show their work. "Someone else should be able to look at your work and understand what you did to solve it" (p. 11).

In a full-class discussion following this activity, students share their problem-solving strategies while the teacher helps deepen their understanding by posing ques-

tions. The teacher also asks students to model a solution suggested by a classmate—such as using the cubes or hundreds chart as shown in the students' work samples. Students with disabilities may need to actually use cards as models to help them connect to the problem situation. Other students can then be asked to try the strategy. Poll students to see who also used a similar approach to give them ownership while you assess students' development. Before moving on, you can discuss strategies not already presented. Then carefully connect to the symbolic representation through writing the equation for the problem. Talk about how this can be linked to an addition story using the same numbers.

Take time to examine the two student work samples. Can you come to a more

## Technology Ideas

Infusing technological tools is important in learning mathematics, as you will learn in Chapter 7. We have infused technology ideas throughout Section II. An icon is used to identify those places within the text or activity where a technology idea or resource is discussed. Descriptions include open-source (free) software, applets, and other Web-based resources, as well as calculator ideas.

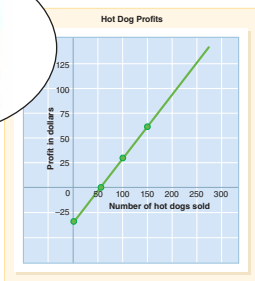


FIGURE 14.25 A graph showing profit as a function of hot dogs sold.

**Verbal Description**  
The amount of profit that can be made selling hot dogs is a function of the number of hot dogs that are sold.

**Context**  
Hot dogs

**Table**

H	P
0	-35
100	25
200	75

**Graph**  
P  
H

**Symbols**  
 $P = 0.65H - 35$

FIGURE 14.26 Five different representations of a function. For any given function, students should see that all these representations are connected and illustrate the same relationship.

## Formative Assessment Notes

Assessment should be an integral part of instruction. Similarly, it makes sense to think about what to be listening for (assessing) as you read about different areas of content development. Throughout the content chapters, you will see Formative Assessment Note icons indicating a short description of ways to assess the topic in that section. Reading these assessment notes as you read the text can also help you understand how best to help your students.



The hot dog problem is a good performance assessment. A good question (which can be adapted to any task) is, "Can you show me where in the table, the graph, and the equation you can find the profit for selling 225 hot dogs?"

The seventh-grade *Connected Mathematics Project (CMP II)* has an entire unit titled "Variables and Patterns," in which students explore and use different representations of functions in real contexts. The lesson shown here focuses on tables and graphs.

**Algebraic Thinking Across the Curriculum**  
One reason the phrase "algebraic thinking" is used instead of "algebra" is that the practice of looking for patterns, regularity, and generalizations goes beyond curriculum topics that are usually categorized as algebra topics. You have already experienced some of this integration—looking at geometric growing patterns and working with perimeter and area. In fact, in *Curriculum Focal Points* (NCTM, 2006), many of the focal points that include algebra connect it to other content areas. In the sections that follow, the emphasis of the content moves to the other content areas, with algebraic thinking used as a tool for discovery. This brief discussion will be developed more fully in later chapters.

**Measurement and Algebra.** Soares, Blanton, and Kaput (2006) describe how to "algebraify" the elementary curriculum. One measurement example they give uses the children's book *Spaghetti and Meatballs for All* (by Burns and Tilley), looking at the increasing number of chairs needed given the growing number of tables.  
Geometric formulas relate various dimensions, areas, and volumes of shapes. Each of these formulas involves at least one functional relationship. Consider any familiar formula for measuring a geometric shape. For example, the circumference of a circle is  $c = 2\pi r$ . The radius is the independent variable, and circumference is the dependent



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